Inflation in curved spacetimes

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1 Introduction

This thesis is written in order to provide a general view of inflation when having curved FRW universes. Inflation is considered now to be a part of the Standard Cosmological Model. Inflation is assumed to begin at about $10^{-36}\text{sec}$ (a little after Planck time $10^{-43}\text{sec}$) and ends at $10^{-34}\text{sec}$. In addition, the beginning of inflation is the time limit after which the physics is, in a good approximation, well understood. Nowadays there is no consistent theory that can explain the physics during Planck era. Nonetheless, String Theory seems to be the theory that could give us the solution, but this will become clear in the future. Returning now to inflation, in this thesis, we solve Friedmann equations for both negatively and positively curved universe and then compare the solutions with the trivial case of a flat universe. Our main purpose is to see how the curvature influences the expansion during inflation and discuss deviations from the Euclidean space. Also, we compute some useful cosmological quantities when horizon crossing occurs, during inflation. After that, we 'count' the number of $e-folds$ contained between the time when a specific length crosses the horizon until the end of inflation. Then we discuss the result and make some conclusions. To end this introduction, I would like to thank my Supervisor Mr Christos Tsagkas for the guidance and, in general, for the whole cooperation.
2 Inflation

2.1 Inflation Idea

Inflation is an era of the early universe during which the last was dominated by quantum vacuum, its scale had a tremendous expansion and after that, the large scale behavior of it had more smoothly changes. At the beginning of inflation the universe had a phase transition and a spontaneously symmetry braking occurred. The inflation model was introduced at 1980’s from Alan Guth, in order to give solution to basic problems of the Standard Big Bang Cosmological Model. These problems were the horizon problem, the flatness problem, as well as the magnetic monopoles problem. This was the old inflation and as it is believed now, it is not accurate and it cannot explain the basic physics, occurred during inflation. The new inflation, as it is called, which is based on ideas mainly of Linde et al, is what gives the right inflation mechanism. According to this scenario, the inflaton field must have a slowly varying region and the effective potential $V(\phi)$ of the $\phi$ field is approximately a constant during inflation.

2.2 Inflation Era

Inflation era is an era after Planck time ($10^{-44}$ sec-Quantum Gravity) during which the dimensions of the universe experienced a dramatic increase. This increase is model depending on the most common of them, predicting an expansion of $10^{43}$ in the scale factor. In order to describe quantum vacuum and inflation evolution we assume that a scalar field $\phi$ exists, the so-called inflation field. Inflation is driven by this inflation field $\phi$. As we shall see, one can derive the pressure and density of this field via the energy-momentum tensor $T^{ab}$ if dealing the quantum vacuum as a perfect fluid.

2.3 Quantum Vacuum

Quantum mechanics has changed our point of view about the vacuum. According to Heisenberg’s Uncertainty Principle, it is possible to borrow an amount of energy $\Delta E$, from nothing, which must be returned in time shorter than $\Delta t$. There is no violation if the process is done in time $\Delta t \leq \frac{\hbar}{\Delta E}$. So, for $\Delta E$ and $\Delta t$ we have, approximately,

$$\Delta E \Delta t \approx \hbar$$

During this period a particle-antiparticle pair with total mass $2m$ and related energy $2mc^2$ can be produced. Thus, for different time intervals $\Delta t$ a variety of particle-antiparticle pairs with masses

$$m \approx \frac{\hbar}{c^2 \Delta t}$$
are created. These moving particles carry charges and as a result Electromagnetic waves are created. It is generally assumed, in Quantum Field theory, that each fundamental particle is associated with a field, or in other words the description of particles is achieved by the field concept. More specifically, spinless particles are described via scalar fields $\phi(x)$, spin-$1/2$ particles described by spinors $\psi(x)$ and the description of spin-1 particles is achieved by using vector fields $A_\mu(x)$. Having the Lagrangian density of a field we can determine the ground state (vacuum) of the theory and the particles are excitations of the vacuum of the theory.

2.4 Inflaton Scalar Field - $\phi$

In order to study the vacuum, we consider that a scalar field $\phi$ exists, which describes the pressure and density of quantum vacuum, and therefore the nature of vacuum itself. One might say that the scalar field $\phi$ exists to provide vacuum energy. First of all we need to express the pressure and density of quantum vacuum as functions of the inflaton field $\phi$. We consider a pseudo-Riemannian spacetime filled with inflaton field $\phi$, which is minimally coupled to gravity. Then, the Lagrangian density of the field will be

$$L_\phi = -\sqrt{-g} \left[ \frac{1}{2} \nabla_a \phi \nabla^a \phi + V(\phi) \right]$$

(1)

where $\phi$ is the inflaton field, $g$ is the determinant of the spacetime metric tensor and $V(\phi)$ is the effective potential which describes the self-interaction of inflaton field. From the above Lagrangian density we can derive the associated stress-energy tensor, which is given by the expression

$$T_{ab}^{(\phi)} = \nabla_a \phi \nabla_b \phi - \left[ \frac{1}{2} \nabla_c \phi \nabla^c \phi + V(\phi) \right] g_{ab}$$

(2)

Now, to achieve a 1+3 covariant fluid-description of the inflaton field $\phi$, we assign a 4-velocity vector to $\phi$ itself. Assuming that the 4-vector $\nabla_a \phi$ is timelike $(\nabla_a \phi \nabla^a \phi < 0)$ over an open spacetime region, the last defines the normals to the spacetime hypersurfaces $\phi(x^a) = \text{constant}$ and the 4-velocity field defined via

$$u_a = -\frac{1}{\dot{\phi}} \nabla_a \phi,$$

(3)

where $\dot{\phi} = u^a \nabla_a \phi \neq 0$, and also the equation $u_a u^a = -1$ (timelike) holds as required. The flow vector $u_a$ defines the time direction and supplies a unique spacetime splitting into space and time. The 3-space orthogonal to $u_a$ has a metric which is given by the projection tensor

$$h_{ab} = g_{ab} + \frac{1}{\dot{\phi}^2} \nabla_a \phi \nabla_b \phi$$

(4)

Also, via the projection tensor, the covariant derivative operator can be defined by

$$D_a = h^b_a \nabla_b$$

(5)
which is orthogonal to \( u_a \) such that the equation \( D_a \phi = 0 \) holds for each \( \phi \) in general. Now, as has been mentioned we treat quantum vacuum as a perfect fluid, thus the energy-momentum tensor will have the form

\[
T_{ab}^{(\phi)} = \rho^{(\phi)} u_a u_b + p^{(\phi)} h_{ab}
\]

where \( \rho^{(\phi)} \) and \( p^{(\phi)} \) are the vacuum density and pressure respectively. We are now able to express \( \rho^{(\phi)} \) and \( p^{(\phi)} \) as functions of inflation field. In addition, the equations

\[
\dot{\phi} = u^a \nabla_a \phi
\]

\[
\nabla_a \phi \nabla^a \phi = -\dot{\phi}^2
\]

hold. The projection tensor \( h_{ab} \) can also be expressed as

\[
g_{ab} = h_{ab} - u_a u_b
\]

Using (3), (7), (8) and (9), the equation (2) reaches

\[
T_{ab}^{(\phi)} = \frac{\dot{\phi}^2}{2} u_a u_b - \frac{1}{2} (-\dot{\phi}^2)(h_{ab} - u_a u_b) - V(\phi)(h_{ab} - u_a u_b)
\]

\[
= \frac{\dot{\phi}^2}{2} u_a u_b + \frac{1}{2} \dot{\phi}^2 h_{ab} - \frac{1}{2} \dot{\phi}^2 u_a u_b - V(\phi) h_{ab} + V(\phi) u_a u_b \Rightarrow
\]

\[
T_{ab}^{(\phi)} = \left[ \frac{\dot{\phi}^2}{2} + V(\phi) \right] u_a u_b + \left[ \frac{\dot{\phi}^2}{2} - V(\phi) \right] h_{ab}
\]

Comparing the last equation with the energy-momentum tensor for a perfect fluid (eq. 3), we derive the vacuum’s density and pressure, which are

\[
\rho^{(\phi)} = \frac{\dot{\phi}^2}{2} + V(\phi)
\]

\[
p^{(\phi)} = \frac{\dot{\phi}^2}{2} - V(\phi)
\]

respectively. From the last equation it is obvious that when the term \( V(\phi) \) is greater than \( \frac{\dot{\phi}^2}{2} \), it is possible to have negative pressure. Negative pressure acts like a repulsive gravitational force and drives the universe in an accelerated expansion. The above conclusions will be clear in next chapters when we will derive the solutions to the Friedmann’s equations during inflation.

2.5 Slowly Varying Scalar Field and State Equation of Q.V.

During inflation the scalar field is slowly varying such that the ‘kinetic’ term \( \frac{\dot{\phi}^2}{2} \) is negligible comparing with the potential \( V(\phi) \) and the last is approximately constant. Then, one can
write
\[ \frac{\dot{\phi}^2}{2} \pm V(\phi) \approx \pm V(\phi) \quad (13) \]

Using the last relation the equations (11) and (12) become
\[ \rho \approx V(\phi) \quad (14) \]
\[ p \approx -V(\phi) \quad (15) \]
that is,
\[ p \approx -\rho = -\rho_0 \quad (16) \]
The last term is approximately a constant, let’s say \( \rho_0 \). The equation \( p = -\rho_0 \) is the state equation of the vacuum during inflation and it will be used in order to solve the Friedmann equations in that era.

### 3 Friedmann Cosmological Models

The isotropy of Cosmic Microwave Background Radiation (CMB) along with the assumption that the Cosmological Principle is correct (The Cosmological Principle states that our position in the universe has nothing special) lead us to the statement that the universe is homogenous and isotropic when observed from large distances. Thus, in the description of the universe we must take into account the above statement. As a result, we do believe that the universe is described (with some approximation) via the simplest cosmological solutions of Einstein Field Equations, the so-called Friedmann equations or sometimes called FRW-models (Friedmann-Robertson-Walker).

#### 3.1 Robertson-Walker linear element

The distance between two neighbouring points lie on the 4-D pseudo-Riemannian manifold, or equivalently the separation between two facts of the spacetime is determined via the metric tensor which, in general, is represented by the 4x4-matrix, with elements depending on the space coordinates \( \{x^a\} \), \( g_{ab} = g_{ab}(x^c) \), with \( a, b, c = 0, 1, 2, 3 \). The linear element in general is given by
\[ ds^2 = g_{ab}dx^a dx^b \quad (17) \]
or
\[ ds^2 = g_00 dt^2 + 2g_{0k}dt dx^k + g_{ij}dx^i dx^j \quad (18) \]
where \( i, j, k = 1, 2, 3 \) and \( (x^0, x^1, x^2, x^3) = (t, x, y, z) \). In FRW models the 3-D spatial space is homogenous and isotropic. Thus, the components \( g_{0k} \) must obey
\[ g_{0k} = 0 \quad (19) \]
with $k = 1, 2, 3$. If we choose $g_{00} = -1$ the above linear element is written as

$$ds^2 = -dt^2 + dl^2$$

(20)

Because of the spherical symmetry of the problem, by using spherical coordinates we have

$$dl^2 = a^2 \left[(1 - Kr^2)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\right]$$

(21)

where $a$ is a function of time ($a = a(t)$), the known scale factor, and the parameter $K$ takes the values $0, \pm 1$ and determines the space geometry. Summarizing, the Robertson-Walker linear element will be

$$ds^2 = -dt^2 + a^2 \left[(1 - Kr^2)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\right]$$

(22)

In this case the metric tensor is $g_{ab} = \text{diag}(-1, a^2(1 - Kr^2)^{-1}, a^2r^2, a^2r^2\sin^2 \theta)$ Depending on the value of $K$ one can have different geometries and as a consequence three types of universes such that $K = 0$ Euclidean geometry (flat universe), $K = -1$ hyperbolic geometry (open universe) and for $K = +1$ spherical geometry (closed universe).

3.2 Friedmann Equations

From Einstein’s field equations and using the assumptions of FRW models, in other words demanding only perfect fluid presence and using the Robertson-Walker metric, we derive the Friedmann’s equations which are (without a cosmological constant)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3} k \rho - \frac{K}{a^2}$$

(23)

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -\frac{1}{2} k (\rho + p) + \frac{K}{a^2}$$

(24)

where $a$ is the scale factor, $\rho$ and $p$ fluid density and pressure respectively and $k = 8\pi G$.\footnote{From now on we will be using natural units, that is $k = 8\pi G = 1 = \hbar = c$}

We can also derive a third equation which it is not independent from the above. Combining the last equations we obtain

$$\frac{\ddot{a}}{a} = -\frac{1}{6} k (\rho + 3p)$$

(25)

which is known as Raychaudhuri equation.

3.3 Density parameter

Introducing the Hubble parameter via $H = \dot{a}/a$ the first one of the Friedmann’s equation gives

$$H^2 = \frac{1}{3} \rho - \frac{K}{a^2}$$

(26)
When having a flat universe \( K = 0 \) that last leads to

\[
H^2 = \frac{1}{3} \rho_c \tag{27}
\]

The density parameter is defined by the ratio

\[
\Omega = \frac{\rho}{\rho_c} \tag{28}
\]

or by substituting \( \rho_c \) from (27) into (28)

\[
\Omega = \frac{\rho}{3H^2} \tag{29}
\]

Thus, the density parameter is defined to be the ratio of the density of the universe over the critical density \( (\rho_c) \) for which is flat. Furthermore, it is obvious that the value \( \Omega = 1 \) corresponds to a flat universe while \( \Omega > 1 \) and \( \Omega < 1 \) are associated with a closed and an open universe respectively.

### 3.4 Basic cosmological scales

Now we have to define in brief some fundamental quantities of cosmology with physical significance. We start by introducing the curvature scale via

\[
\lambda_K = \frac{a}{|K|} \tag{30}
\]

and \( \lambda_K \) is the limit beyond which one must take into account phenomena associated with curvature. As can be easily seen, in a flat universe \( (K = 0) \) there are no curvature phenomena (the scale in which they observed is \( \lambda_K \rightarrow \infty \)) while in curved universes \( (|K| = 1) \) the curvature cannot be ignored at distances greater than \( a \). The fact that the speed of light \( c \) is a constant along with the belief that the age of the universe is finite, define a specific distance such that the points whose distance is greater than that, are unable to interact with each other and to pass information. This distance is

\[
\lambda_p = ar \tag{31}
\]

where \( r \) is the distance a photon has traveled until time \( t \). Another quantity is the Hubble’s radius or sometimes called Hubble horizon which gives an approximation for the time required for physical phenomena to develop and is given by

\[
\lambda_H = \frac{1}{H} \tag{32}
\]

The last quantity which is associated with length and will be used below is the well-known wavelength. The evolution of the universe and in particular the scale factor evolution influences all the physical magnitudes depending on length. Thus, because of the expansion
(or contraction) of the universe all the wavelengths must develop proportional to the scale factor. Consider a wavelength $\lambda_m$, this is related linearly to the scale factor by the relation

$$\lambda_m = \frac{a}{m}$$  \hspace{1cm} (33)

where $m^2$ is the wavenumber associated with the given length.

### 3.5 Conformal time-$\eta$

It is convenient sometimes to have solutions of Friedmann’s equations in parametric form. We introduce a quantity called conformal time which is defined by

$$\frac{d\eta}{dt} = \frac{1}{a}$$  \hspace{1cm} (34)

where $a$ is the scale factor, $t$ is the proper time, and we can express the scale factor and the proper time as functions of $\eta$ in the doublet $\{a(\eta), t(\eta)\}$. Thus, we can have the scale factor evolution in parametric form with the conformal time being the parameter.

### 4 Solutions of Friedmann’s equation during inflation

In this section we are going to solve Friedmann’s equation during inflation in curved space-times ($K = \pm 1$). We also give the trivial solution of Euclidean geometry ($K = 0$) just to compare with the curved models and see how the curvature influences the expansion.

#### 4.1 $K = 0$ solution

From the first of Friedmann’s equations (23) for $K = 0$ and using $\rho = \rho_o = constant$ it follows that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho_o$$  \hspace{1cm} (35)

Assuming that for $t = t_o$ is $a(t_o) = a_o$ we instantly take

$$a(t) = a_o \exp \sqrt{\frac{\rho_o}{3}} (t - t_o)$$  \hspace{1cm} (36)

2 We are using m for wavenumber rather k in order to avoid confusion with other indices
Also from (26) one can easily see that the Hubble parameter is a constant during inflation (only in flat case $K = 0$) $H^2 = H_o^2 = \frac{\rho_o}{3}$, and the above equation can be written as

$$a(t) = a_o \exp H_o(t - t_o)$$

and one has an exponential increase of the dimensions of the universe.

### 4.2 $K = \pm 1$ solutions

Now we are going to find an expression for the scale factor in curved models. From the first of Friedman’s equation and recalling the state equation of the vacuum during inflation $p = -\rho_o$ we obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}k\rho_o - \frac{K}{a^2}$$

or, by setting $b = \frac{\rho_o}{3} > 0$ and multiplying each side of equation with $a^2$

$$\dot{a}^2 = ba^2 - K$$

and because at inflation era we have accelerated expansion $\dot{a} > 0$,is

$$\dot{a} = \sqrt{ba^2 - K} \Rightarrow$$

$$\int \frac{da}{\sqrt{ba^2 - K}} = \int dt$$

where $b = \rho_o/3$

#### 4.2.1 $K = -1$

For $K = -1$ the last equation is written as

$$\int \frac{da}{\sqrt{ba^2 - 1}} = \int dt$$

and if considering the change of variable $\sqrt{ba} = \cosh u$ is integrated immediately to give \(^3\)

$$a(t) = \frac{1}{\sqrt{b}} \cosh \sqrt{b}t + c$$

where $c$ is arbitrary constant. Now, assuming that at time $t_o$ is $a(t_o) = a_o$, and using the hyperbolic functions identities, $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$ and $\cosh^2 x - \sinh^2 x = 1$ we derive

$$a(t) = a_o \left[ \cosh \sqrt{b}(t - t_o) + \left(1 + \frac{1}{ba^2} + 1\right) \sinh \sqrt{b}(t - t_o) \right]$$

\(^3\) for an exact computation see Appendix
which gives the scale factor evolution on an open $K = -1$ universe during inflation. Obviously the scale factor can also be expressed in terms of the exponential functions $e^{\pm \sqrt{b}(t-t_o)}$.

4.2.2 $K = +1$

For $K = +1$ we have to solve
\[
\int \frac{da}{\sqrt{ba^2 + 1}} = \int dt
\]
which is solved quite straightforward. As previously but now considering the transformation $\sqrt{ba} = \sinh u$ and again assuming that for $t = t_o$ is $a(t_o) = a_o$ we, finally, obtain
\[
a(t) = a_o \left[ \cosh \sqrt{b}(t - t_o) + \left(1 + \sqrt{\frac{1}{ba_o^2} - 1}\right) \sinh \sqrt{b}(t - t_o) \right]
\]
and we observe that the only difference with the case $K = -1$ is the minus sign inside the square root. Thus, we can generalize and write
\[
a(t) = a_o \left[ \cosh \sqrt{b}(t - t_o) + \left(1 + \sqrt{\frac{1}{ba_o^2} - K}\right) \sinh \sqrt{b}(t - t_o) \right]
\]
with $K = -1$ and $K = +1$ corresponds to an open and a closed universe respectively. We can also find the solutions in another way. Alternatively, using the third of Friedmann’s equations, or as has been mentioned Raychaudhuri’s equation (25)
\[
\ddot{a} - ba = 0
\]
and because $b = \rho_o/3 > 0$ the general solution will be
\[
a(t) = Ae^{\sqrt{b}t} + Be^{-\sqrt{b}t}
\]
where $A,B$ are arbitrary constants, and from this equation it seems that the evolution of the scale factor is independent of the curvature $K$. Of course, this is not true because the curvature dependence is seeking in constants $A,B$. It is easy to see that, taking the first derivative of the above function
\[
\dot{a} = \sqrt{b}(Ae^{\sqrt{b}t} - Be^{-\sqrt{b}t})
\]
and by substituting the last two equations into (39) it follows that
\[
AB = \frac{K}{4b}
\]
from which the relation between the constants $A$, $B$ and curvature index $K$ is shown. Now, same as before if we assume that for $t = t_o$ is $a(t_o) = a_o$ and use the last equation, finally,
we get

\[ a(t) = \frac{a_o}{2} \left[ \left( 1 + \sqrt{1 + \frac{1}{ba_o^2}} \right) e^{\sqrt{b}(t-t_o)} + \left( 1 - \sqrt{1 + \frac{1}{ba_o^2}} \right) e^{-\sqrt{b}(t-t_o)} \right] \]  

(51)

and,

\[ a(t) = \frac{a_o}{2} \left[ \left( 1 + \sqrt{1 + \frac{1}{ba_o^2}} \right) e^{\sqrt{b}(t-t_o)} + \left( 1 - \sqrt{1 + \frac{1}{ba_o^2}} \right) e^{-\sqrt{b}(t-t_o)} \right] \]

(52)

corresponding to cases \( K = -1 \) and \( K = +1 \) respectively. Again, we can generalize the aboves to the equation

\[ a(t) = \frac{a_o}{2} \left[ \left( 1 + \sqrt{1 - \frac{K}{\Omega_o}} \right) e^{\sqrt{\Omega_o}(t-t_o)} + \left( 1 - \sqrt{1 - \frac{K}{\Omega_o}} \right) e^{-\sqrt{\Omega_o}(t-t_o)} \right] \]

(53)

and recall that \( b = \rho_o/3 \)

### 4.3 Relation with observable quantities and asymptotic behavior of scale factor

It is important to express the scale factor in terms of Hubble’s and density’s parameters at the time \( t = t_o \), which are denoted as \( H_o \) and \( \Omega_o \) respectively. In order to achieve that we rewrite the definition of \( \Omega \) at time \( t_o \), it holds that

\[ \Omega_o = \frac{\rho}{3H_o^2} \]

(54)

but during inflation is \( \rho = \rho_o = constant \) and the last is written as

\[ \Omega_o = \frac{\rho_o}{3H_o^2} = \frac{b}{H_o^2} \]

or

\[ \sqrt{b} = \sqrt{\Omega_o}H_o \]

(55)

and the combination of the last equation together with (53) gives

\[ a(t) = \frac{a_o}{2} \left[ \left( 1 + \sqrt{1 - \frac{K}{\Omega_o}} \right) e^{\sqrt{\Omega_o}(t-t_o)} + \left( 1 - \sqrt{1 - \frac{K}{\Omega_o}} \right) e^{-\sqrt{\Omega_o}(t-t_o)} \right] \]

(56)

and if we notice that

\[ H_o^2(\Omega_o - 1) = \frac{K}{a_o^2} \]

(57)

the last can be reduced in a more attractive form

\[ a(t) = \frac{a_o}{2} \left[ \left( 1 + \frac{1}{\sqrt{\Omega_o}} \right) e^{\sqrt{\Omega_o}(t-t_o)} + \left( 1 - \frac{1}{\sqrt{\Omega_o}} \right) e^{-\sqrt{\Omega_o}(t-t_o)} \right] \]

(58)

where the curvature dependence (\( K \)) is seeking in the expression for \( \Omega_o \). Now, it is important to see how the curvature affects scale factor’s evolution and also, the consequences on Hubble’s parameter. In order to achieve that, we will be considering in both periods at the beginning and at the end of inflation epoch.
4.3.1 Beginning of inflation

We are going to study the impact of curvature at the very beginning of inflation. Setting \( x = \sqrt{\Omega_o H_o (t - t_o)} \) and \( \mu = \sqrt{1 - \frac{K}{\Omega_o H_o a_o^2}} \) the last equation can be written in the more compact form

\[
a(t) = \frac{a_o}{2} \left[ (1 + \mu) e^x + (1 - \mu) e^{-x} \right]
\]

At the beginning of inflation \( t - t_o \to 0 \) and \( x \to 0 \), so by using a Taylor expansion around \( x = 0 \) we can omit terms higher then linear. Thus, one has

\[
e^x \approx 1 + x
\]

as well as,

\[
e^{-x} \approx 1 - x
\]

and by substituting the last ones in the above equation for scale factor, after a little algebra we obtain

\[
a(t) = a_o (1 + \mu x)
\]

also, it holds that

\[
\mu x = \sqrt{1 - \frac{K}{\Omega_o H_o^2 a_o^2}} \sqrt{\Omega_o H_o (t - t_o)} = \sqrt{\Omega_o H_o^2 - \frac{K}{a_o^2}}(t - t_o) \Rightarrow \mu x = \sqrt{b - \frac{K}{a_o^2}}(t - t_o)
\]

but from Friedmann’s equation for \( t = t_o \) it holds that \( H_o^2 = b - \frac{K}{a_o^2} \) and thus, we have

\[
\mu x = H_o (t - t_o)
\]

returning to the scale factor expression now, it follows that

\[
a(t) = a_o (1 - H_o t_o) + a_o H_o t
\]

The last equation gives us a very significant result, in both curvature cases (\( K = \pm 1 \)): the curvature of the universe influences strongly enough scale factor’s behavior such that the last has no exponential expansion as in the flat case (\( K = 0 \)), but it grows linearly with time at the beginning of inflation. Notice that the last result is independent of the value of \( K \) when \( K \neq 0 \). The first derivative of \( a \) with respect to \( t \) will be \( \dot{a} = a_o H_o \) and the Hubble parameter for small \( x \) is

\[
H = \frac{\dot{a}}{a} \approx \frac{1}{(1 - H_o t_o) + t}
\]

which is not a constant. This result is also in contrast with the behavior of Hubble’s parameter when there is no curvature (\( K = 0 \)) where the last remains constant at all times during inflation.
4.3.2 Ending of inflation

Now, we are going to see what happens at the end of inflation i.e. when $t >> t_o \Rightarrow t - t_o \approx t$ and one has

$$e^x >> e^{-x} \quad (67)$$

or

$$e^x \pm e^{-x} \approx e^x \quad (68)$$

and we can neglect the term $e^{-x}$ in the expression of the scale factor. Using the last approximation we derive for the scale factor

$$a(t) = \frac{a_o}{2} (1 + \mu) e^x \Rightarrow$$

$$a(t) = \frac{a_o}{2} (1 + \frac{H_o}{\sqrt{b}}) e^{\sqrt{b}(t-t_o)} \quad (69)$$

And one can easily see that at the end of inflation the curvature does not influence scale’s factor evolution with the last growing exponentially with time as in the Euclidean case. Thus, independent of the initial geometry of the universe ($K = \pm$) the last when inflation ends has become extremely flat and one has the well-known exponential expansion as in the flat case. To summarize, we can say that in the beginning of inflation the curvature of the universe do have an important impact on the expansion of the universe and slows it down such that the scale factor grows linearly with time. Though, when inflation ends the curvature is no longer significant with the scale factor growing exponentially with time.

5 Conformal time solutions

As has been mentioned earlier, one can have solutions of FRW models in parametric form, with the conformal time $\eta$ being the parameter. In this section we will give the solutions in parametric form during inflation and see scale factor’s and time behavior as functions of conformal time $\eta$. In what follows, at the beginning of inflation will be $\eta \to -\infty$ while at the end will be $n \to 0$ for an open model $K = -1$. In the positively curved model ($K = +1$), $\eta$ take values in the interval $(-\frac{\pi}{2}, 0)$

5.1 $K = 0$ solution

We had introduced conformal time via $\frac{dn}{dt} = \frac{1}{a}$. From the chain rule we have

$$\dot{a} = \frac{da}{dt} = \frac{da}{dn} \frac{dn}{dt} = \frac{a'}{a} \quad (70)$$
where the prime denotes differentiation with respect to \( \eta \). Substituting the last into the first of Friedmann’s equation for \( K = 0 \) (eq. 35) we obtain

\[
a' = \sqrt{b}a^2 \Rightarrow \frac{da}{a^2} = \sqrt{b}dn \Rightarrow
\]

\[
a = \frac{1}{-\sqrt{b}\eta + C}
\]

(71)

where \( C \) is an arbitrary constant. Now, if we assume that at the end of inflation \( (\eta \to 0) \) is \( a \gg 1 \), from the last equation we have that \( C \to 0 \). Thus, one has

\[
a = \frac{1}{-\sqrt{b}\eta}
\]

(72)

and the time will be given by

\[
\frac{d\eta}{dt} = \frac{1}{a} \Rightarrow t = \int adn
\]

\[
t = -\frac{1}{\sqrt{b}} \ln|n| = -\frac{1}{\sqrt{b}} \ln(-n)
\]

(73)

because of \( \eta < 0 \Rightarrow |\eta| = -\eta \). Thus for the trivial case \( (K=0) \) we have the doublet

\[
\left\{ a = \frac{1}{-\sqrt{b}\eta}, t = -\frac{1}{\sqrt{b}} \ln(-n) \right\}
\]

(74)

describing scale factor evolution and \( \eta \in (-\infty, 0) \)

5.2 \( K = -1 \)

If we use the relation \( \dot{a} = \frac{a'}{a} \) the Friedmann equation in terms of conformal time takes the form

\[
\left( \frac{a'}{a^2} \right)^2 = b - \frac{K}{a^2} \Rightarrow a' = a\sqrt{ba^2 - K}
\]

(75)

Thus, in order to find the solution in a negatively curved universe we have to solve

\[
a' = a\sqrt{ba^2 + 1} \Rightarrow \frac{da}{a\sqrt{ba^2 + 1}} = d\eta
\]

(76)

which is integrated quite straightforward, to give\(^4\)

\[
a(\eta) = \frac{1}{\sqrt{b} \sinh (-\eta + C)}
\]

(77)

\(^4\) The integration is given at the Appendix
where \( C \) is an arbitrary constant. If we consider that at the end of inflation \((\eta \to 0)\) is \( a \gg 1 \) then we have
\[
\frac{1}{\sin C} \to \infty \Rightarrow C \to 0
\] (78)
such that
\[
a(\eta) = \frac{1}{\sqrt{b} \sinh (-\eta)}
\] (79)
with \( \eta \in (-\infty, 0) \), and one can determine \( t(\eta) \) via
\[
dt = adn \Rightarrow t = \int \frac{1}{\sqrt{b} \sinh (-\eta)} \, d\eta
\] (80)
and after the integration we obtain
\[
t(\eta) = \frac{1}{\sqrt{b}} \ln \left[ \frac{1 + \cosh (-\eta)}{\sinh (-\eta)} \right]
\] (81)
Summarizing the last results, the parametric form of scale factor, in an open universe at inflation epoch, is given by the doublet
\[
\left\{ a(\eta) = \frac{1}{\sqrt{b} \sinh (-\eta)}, t(\eta) = \frac{1}{\sqrt{b}} \ln \left[ \frac{1 + \cosh (-\eta)}{\sinh (-\eta)} \right] \right\}
\] (82)
and recall that when dealing with open universe \((K = -1)\) \( \eta \in (-\infty, 0) \) such that \( a > 0 \) and \( t > 0 \) always hold. As we shall see later these relations will be found quite convenient when we will be about to determine the horizon crossing. Our next step is to determine the parametric solutions for a positively curved universe and then compare the solutions.

5.3 \( K = +1 \) solution

Now, to close the conformal time solutions we give the solution for a positively curved universe (spherical geometry). In this case the differential equation will be
\[
a' = a\sqrt{ba^2 - 1} \Rightarrow \frac{da}{a\sqrt{ba^2 - 1}} = d\eta
\] (83)
and after an integration it follows that
\[
a(\eta) = \frac{1}{\sqrt{b} \sin (-\eta + C)}
\] (84)
and again assuming that at the end of inflation epoch \((\eta \to 0)\) is \( a \gg 1 \) and as a result
\[
C \to 0
\] (85)
\footnote{Same as the above, this integral is computed in the Appendix}
such that
\[ a(\eta) = \frac{1}{\sqrt{b} \sin(-\eta)} \] (86)
and \( t \) is determined via
\[
t = \int adn \Rightarrow t = \frac{1}{\sqrt{b}} \int \frac{1}{\sin(-\eta)} d\eta \Rightarrow \\
\frac{1}{\sqrt{b}} \int \frac{\sin(-\eta)}{\sin^2(-\eta)} d\eta = \frac{1}{\sqrt{b}} \int \frac{d\cos(-\eta)}{\sin^2(-\eta)}
\]
and under the variable transformation \( u = \sin(-\eta) \)
\[
t = \frac{1}{\sqrt{b}} \int \frac{du}{1-u^2} = \frac{1}{\sqrt{b}} \int \frac{du}{(1-u)(1+u)} \Rightarrow \\
t = \frac{1}{2\sqrt{b}} \ln \left( \frac{1+u}{1-u} \right) \Rightarrow t(\eta) = \frac{1}{2\sqrt{b}} \ln \left( \frac{1+\cos(-\eta)}{1-\cos(-\eta)} \right)
\] (87)
or equally
\[
t(\eta) = \frac{1}{\sqrt{b}} \ln \left( \frac{1+\cos(-\eta)}{\sin(-\eta)} \right)
\] (88)
and the doublet which gives us the parametric solution for an open universe is now
\[
\{ a(\eta) = \frac{1}{\sqrt{b} \sin(-\eta)}, t(\eta) = \frac{1}{\sqrt{b}} \ln \left( \frac{1+\cos(-\eta)}{\sin(-\eta)} \right) \}
\] (89)
with \( \eta \in (-\frac{\pi}{2}, 0) \) as has been mentioned and \( \sin(-\eta) > 0 \) always as required for the quantity inside the logarithm to be positive.

5.4 Generalization of \( K = \pm 1 \) solutions

Often, in science it is desired , if it is possible, to generalize things. In our case now we want to have a general expression for \( a(\eta) \) and obtain the different kinds of solutions by setting \( K = \pm 1 \). Thus, we shall give here a generalization. Using the obvious identity connecting trigonometric and hyperbolic functions \( \sin(i x) = i \sinh(x) \) we observe that (79) and (86) can both be derived via the formula
\[
a(\eta) = \frac{1}{\sqrt{b} i^{1+K}} \frac{1}{\sin(-i^{1+K} \eta)}
\] (90)
by setting \( K = -1 \) and \( K = +1 \) respectively and where \( i = \sqrt{-1} \) is the imaginary unity. The last equation combines the parametric equation of scale factor , for a positively and
negatively curved universe, in one as we claimed. In the same manner but now using the fact
that \( \cos(-ix) = \cosh(-x) \) in addition, one derives

\[
t(\eta) = \frac{1}{\sqrt{b}} \ln \left( \frac{1 + \cos(-i^{1-K} \eta)}{i^{1-K} \sin(-i^{1-K} \eta)} \right)
\]

and again by setting \( K = -1 \) and \( K = +1 \) we have the parametric form of time in an open
and closed universe respectively. Thus, the compact doublet containing both negative and
positive curvature index will be

\[
\begin{align*}
a(\eta) &= \frac{1}{\sqrt{b}} \frac{1}{i^{1-K} \sin(-i^{1-K} \eta)},
t(\eta) &= \frac{1}{\sqrt{b}} \ln \left( \frac{1 + \cos(-i^{1-K} \eta)}{i^{1-K} \sin(-i^{1-K} \eta)} \right)
\end{align*}
\]

from which by setting \( K = -1 \) we derive the solution (82) while for \( K = +1 \) one gets solution
(89). Recall that when we have an open universe, \( \eta \) varies in the interval \((-\infty, 0)\) and for
a closed universe in \((-\pi/2, 0)\) There is a deep reason why we found expressions in terms of
conformal time \( \eta \). As it will be clear at the next chapter parametric solutions will help us to
find some useful quantities, such as the time when a specific wavelength crosses the horizon,
in a quite easy way. Now, the Hubble parameter will be

\[
H = \frac{\dot{a}}{a} = \frac{1}{a^2} \frac{da}{d\eta} \Rightarrow
\]

and after a little algebra

\[
H = \sqrt{b} t^{1-K} \cos(-i^{1-K} \eta)
\]

from which one can see that at the end of inflation \((\eta \to 0)\) is

\[
H \approx \sqrt{b} = \text{constant}
\]

as it was expected.

6 Horizon Crossing

In contrast to the flat case at which the Hubble radius remains constant at all times during
inflation \((\lambda_H = 1/H_o = \text{constant}, \text{for } K = 0)\) when dealing with curvature models things are
different. Hubble parameter as well as Hubble radius are not constants at these cases. Having
a specific perturbation of wavelength \( \lambda_m \), characterized by the wavenumber \( m \) \((\lambda_m = a/m)\),
it is important to know when this length crosses the horizon, during inflation \(^6\), and it
behaves as a classical perturbation. After the horizon crossing we say that the perturbation
has been 'frozen out' and, thus, has become a classical perturbation. In order to find an
expression giving us the time when a specific length crosses the horizon as a function of the

\(^6\) Each scale reenters the horizon after inflation in radiation and matter dominated eras
wavenumber $t_{H,C} = f(m)$ it is convenient at first to play with conformal time or in other words using the parametric scale factor solutions \{a(\eta), t(\eta)\}. Now, demanding that at the beginning $t = t_o$ of inflation all scales are located inside the horizon, we have the inequality

$$\left( \frac{\lambda_m}{\lambda_H} \right)_o < 1 \Rightarrow \frac{a_oH_o}{m} < 1 \Rightarrow m > a_oH_o = \dot{a}_o$$  \hspace{1cm} (95)$$

recalling that

$$\lambda_H = \frac{1}{H}$$  \hspace{1cm} (96)$$
$$\lambda_m = \frac{a}{m}$$  \hspace{1cm} (97)$$
a length crosses the horizon when the equation

$$\left( \frac{\lambda_m}{\lambda_H} \right) = 1$$  \hspace{1cm} (98)$$

holds. Combining the last three equations we derive

$$\frac{aH}{m} = 1 \Rightarrow aH_{H,C} = m$$  \hspace{1cm} (99)$$

and the last equation determines when the length which is described by the wavenumber $m$ crosses the horizon. As has been mentioned before, our first step is to find the value of conformal time when the horizon crossing occurs. In doing so, let us remind that

$$\dot{a} = a' \Rightarrow \dot{a}/a = a_H$$  \hspace{1cm} (100)$$

where the prime denotes derivative with respect to conformal time and from the last two equations we obtain

$$\left( \frac{a'}{a} \right)_{H,C} = m$$  \hspace{1cm} (101)$$

and from the last we obtain $\eta$ at horizon crossing. We are going to find, now, $\eta_{H,C}$ in both $K = \pm 1$ cases.

### 6.0.1 Horizon Crossing for $K=-1$

Expression (79) if we use the identity sinh ($-\eta$) = $\frac{1}{2}(e^{-\eta} - e^{\eta})$ can also be written in the form

$$a(\eta) = \frac{2}{\sqrt{b}} \frac{e^{\eta}}{1 - e^{2\eta}}$$  \hspace{1cm} (102)$$

and by taking the first derivative with respect to $\eta$ we have

$$\frac{d\eta}{d\eta} = \frac{2}{\sqrt{b}} \frac{e^{\eta}(1 + e^{2\eta})}{(1 - e^{2\eta})^2}$$  \hspace{1cm} (103)$$

\[20\]
and by substituting the last two into (101) we derive the equation
\[
\left( \frac{1 + e^{2\eta}}{1 - e^{2\eta}} \right)_{\eta=\eta_{H.C}} = m \Rightarrow 
\]
\[e^{2\eta_{H.C}} = \frac{m - 1}{m + 1} \tag{104}\]
and by taking the logarithm of the above equation we obtain
\[
\eta_{H.C} = \frac{1}{2} \ln \left| \frac{m - 1}{m + 1} \right| \tag{105}\]
In addition, from inequality (95) it follows that
\[
m > \dot{a}_0 = \left( \frac{a'}{a} \right)_{t=t_o} = \left( \frac{1 + e^{2\eta}}{1 - e^{2\eta}} \right)_{t=t_o} \tag{106}\]
but at the beginning of inflation \(t = t_o\) is \(\eta \to -\infty\), such that
\[
\left( \frac{1 + e^{2\eta}}{1 - e^{2\eta}} \right)_{t=t_o} = \lim_{\eta \to -\infty} \left( \frac{1 + e^{2\eta}}{1 - e^{2\eta}} \right) = 1 \tag{107}\]
and as a result
\[
m > 1 \tag{108}\]
always. And because of \(m > 1\) we can take out the absolute value inside the logarithm and finally obtain
\[
\eta_{H.C} = \frac{1}{2} \ln \left( \frac{m - 1}{m + 1} \right) \tag{109}\]
which gives us the value of conformal time when a specific length that has wavenumber \(m\) crosses the horizon. Thus, given a perturbation associated with a length which is characterized by a certain wavenumber we can determine when this scale will cross the horizon and become a classical perturbation in the case \(K = -1\). Now, we are going to find \(\eta_{H.C}\) in a negatively curved universe \((K = +1)\) and then compute the exact time \(t_{H.C}\) when the horizon crossing occurs in both cases.

### 6.0.2 Horizon Crossing for \(K=+1\)

Using the equality \(\sin \left( -\eta \right) = \frac{1}{2i} \left( e^{-i\eta} - e^{i\eta} \right)\) we can rewrite equation (86) in the more convenient form
\[
a(\eta) = \frac{2i}{\sqrt{b}} \frac{e^{i\eta}}{1 - e^{2i\eta}} \tag{110}\]
with first derivative
\[
a' = \frac{da}{d\eta} = \frac{2i}{\sqrt{b}} \frac{ie^{i\eta}(1 + e^{2i\eta})}{(1 - e^{2i\eta})^2} \tag{111}\]
and by substituting into (101) it follows that

\[\frac{e^{i2\eta} + 1}{e^{i2\eta} - 1} = im\]  

(112)

and after a little algebra it can be written as

\[e^{i2\eta} = \frac{1 + im}{1 - im} = -\frac{(1 + im)^2}{1 + m^2} \Rightarrow\]

\[e^{i2\eta_{H.C}} = \frac{m^2 - 1}{m^2 + 1} + i\frac{2m}{m^2 + 1}\]  

(113)

by using Euler’s formula

\[e^{i2\eta_{H.C}} = \cos (2\eta_{H.C}) + i \sin (2\eta_{H.C})\]  

(114)

and comparing the last two equations we derive

\[\cos (2\eta_{H.C}) = \frac{m^2 - 1}{m^2 + 1}\]  

(115)

\[\sin (2\eta_{H.C}) = \frac{2m}{m^2 + 1}\]  

(116)

which ones are not independent from each other. Going one step further we use the trigonometric identities \(\cos^2 x = 1/2(1 + \cos (2x))\) and \(\sin^2 x = 1/2(1 - \cos (2x))\) to finally obtain

\[\cos \eta_{H.C} = \frac{m}{\sqrt{m^2 + 1}}\]  

(117)

\[\sin \eta_{H.C} = \frac{1}{\sqrt{m^2 + 1}}\]  

(118)

where we have used the fact that on a closed universe \(\eta \in (-\pi/2, 0)\). Note that it is not appropriate to have an explicit form for \(\eta_{H.C}\) because the thing which concern us is the value of proper time at horizon crossing. We have only used conformal time to make our life easier. In the next we determine \(t_{H.C}\) as well as some others useful cosmological parameter when a given length crosses the horizon.

### 6.1 Cosmological parameters at horizon crossing

Now it is time to compute what we wanted from the beginning, the value of proper time when the horizon crossing occurs. First we will find that time for \(K = -1\) next for \(K = +1\) and later give some generalizations and determine the value of other parameters of cosmological interest such as Hubble parameter and density parameter at that time.
6.2 Proper time at horizon crossing

6.2.1 $K=-1$

From equation (109) it follows that

$$e^{\eta_{H.C}} = \sqrt{\frac{m-1}{m+1}}$$  \hspace{1cm} (119)

and

$$e^{-\eta_{H.C}} = \sqrt{\frac{m+1}{m-1}}$$  \hspace{1cm} (120)

which are translated into

$$\cosh(-\eta_{H.C}) = \frac{m}{\sqrt{m^2 - 1}}$$  \hspace{1cm} (121)

$$\sinh(-\eta_{H.C}) = \frac{1}{\sqrt{m^2 - 1}}$$  \hspace{1cm} (122)

and by substituting the last two in (81) we derive for the proper time at the horizon crossing

$$t_{H.C} = \frac{1}{\sqrt{b}} \ln\left(m + \sqrt{m^2 - 1}\right)$$  \hspace{1cm} (123)

when having an open $K = -1$ universe.

6.2.2 $K=+1$

In the same manner for a closed universe by substituting equations (117) and (118) into (88) it follows that

$$t_{H.C} = \frac{1}{\sqrt{b}} \ln\left(m + \sqrt{m^2 + 1}\right)$$  \hspace{1cm} (124)

which gives us the time when a scale with a wavenumber $m$ crosses the horizon in the case $K = +1$

6.3 Generalization

As seen before, the generalization of the two above equations is quite straightforward, and reaches

$$t_{H.C} = \frac{1}{\sqrt{b}} \ln\left(m + \sqrt{m^2 + K}\right)$$  \hspace{1cm} (125)

with $K = \pm 1$. The last gives us the time when a specific length crosses the horizon in both negatively and positively curved universes. And from the last it is obvious that for small scales, or in other words big wavenumbers, we have

$$m^2 >> 1 \Rightarrow m^2 + K = m^2 \pm 1 \approx m^2$$  \hspace{1cm} (126)
such that
\[ t_{H.C} \approx \frac{1}{\sqrt{b}} \ln (2m) \] (127)
which tells us that we already expected, small lengths crosses the horizon at the end of inflation and independent of the curvature index \( K \).

### 6.4 \( K=0 \)

Let us now derive an expression for horizon crossing in a flat universe. Again, doing the same steps as before we finally derive
\[ t(\eta_{H.C}) = \frac{1}{\sqrt{b}} \ln (m) \] (128)
which gives us the time when horizon crossing occurs in an Euclidean universe.

### 6.5 A further generalization

Having the expressions for \( t_{H.C} \) in all geometrical cases it is not difficult to generalize, and include all the geometries. The generalization will be
\[ t_{H.C} = \frac{1}{\sqrt{b}} \ln \left( m + \sqrt{m^2 + K + (1 - K)(1 + K)m} \right) \Rightarrow \] (129)
\[ t_{H.C} = \frac{1}{\sqrt{b}} \ln \left( m + \sqrt{m^2 + K + (1 - K^2)m} \right) \] (130)
where, know, \( K = \pm 1, 0 \), and the last reproduces equations (123),(124) as well as (128).

Once we have determine \( t_{H.C} \) let us now find the values of some useful cosmological parameters at horizon crossing. From Friedmann equation we have
\[ H^2 = \frac{b - \frac{K}{a^2}}{a^2} \Rightarrow \Omega - 1 = \frac{K}{a^2} \] (131)
which at the horizon crossing (\( a_{H.C} = m \)) gives
\[ \Omega_{H.C} = 1 + \frac{K}{m^2} \] (132)
and again we see that the big wavenumbers (\( m \gg 1 \)) cross the horizon at the end of the inflation when the universe has become almost flat \( \Omega \approx 1 \). We continue computing the scale factor at horizon crossing at both models. Again from Friedmann equation by multiplying with \( a^2 \) this time, we derive
\[ a^2 = ba^2 - K \] (133)
which at the horizon crossing takes the form

\[ a_{\text{H.C}} = \sqrt{\frac{m^2 + K}{b}} \]  

so for a given length we know the value of scale factor when it crossed out the horizon. The Hubble parameter will be

\[ H_{\text{H.C}} = \left( \frac{\dot{a}}{a} \right)_{\text{H.C}} = \sqrt{b} \frac{m}{\sqrt{m^2 + K}} \]  

which has the behaviour \( H \approx \sqrt{b} \) for big enough wavenumbers. In addition, we can express the proper time when horizon crossing occurs in terms of the density parameter at that time. Indeed, equations (125) and (129) are combined to give

\[ t_{\text{H.C}} = \frac{1}{\sqrt{b}} \ln \left( \frac{1 + \sqrt{\Omega_{\text{H.C}}}}{\sqrt{1 - \Omega_{\text{H.C}}}} \right) \]  

where we have used the fact that \( \sqrt{\frac{K}{\Omega_{\text{H.C}} - 1}} = \frac{1}{\sqrt{1 - \Omega_{\text{H.C}}}} \).

7 Numbering the e-folds

The number \( N \) of e-folds contained inside two times \( t_1, t_2 \) with \( t_2 > t_1 \) at inflation era is defined via

\[ e^N = \frac{a(t_2)}{a(t_1)} \Rightarrow N = \ln \left( \frac{a(t_2)}{a(t_1)} \right) \]  

where \( a(t_1) \) and \( a(t_2) \) are scale’s factor value at times \( t_1 \) and \( t_2 \) respectively. Now, each scale \( \lambda \) living in the universe because of universe’s expansion develops as

\[ \lambda \propto a \]  

where \( a \) is the scale factor. Taking, now, the ratio of scale factor between the end of inflation and the time when a certain length crosses the horizon we derive

\[ \frac{a_f}{a_{\text{H.C}}} = \frac{\lambda_f}{\lambda_{\text{H.C}}} \]  

but, at horizon crossing it holds that

\[ \lambda_{\text{H.C}} = \lambda_H = \sqrt{\frac{3\Omega_{\text{H.C}}}{k\rho_0}} \]
or, using $k = m_{pl}^{-2}$ it follows that

$$\lambda_{H,C} = m_{pl} \sqrt{\frac{3\Omega_{H,C}}{\rho_o}}$$  \hspace{1cm} (140)$$

where $m_{pl}$ is the Planck mass. The number $N$ of $e$ - $folds$ contained between the time when a specific length crosses the horizon until the end of inflation will be

$$e^N = \frac{\lambda_f}{\lambda_{H,C}}$$  \hspace{1cm} (141)$$

where the index f denotes the end of inflation. If we consider that after inflation and until the reheating process the universe is $\phi$ – $dominated$ such that the density obeys

$$\rho \propto \frac{1}{a^3}$$  \hspace{1cm} (142)$$

and if $\rho_o$ and $\rho_{RH}$ represent the density at the end of inflation and at reheating respectively, we have

$$\frac{\rho_o}{\rho_{RH}} = \left(\frac{a_{RH}}{a_f}\right)^3 = \left(\frac{\lambda_{RH}}{\lambda_f}\right)^3 \Rightarrow$$

$$\frac{\lambda_f}{\lambda_{RH}} = \left(\frac{\rho_{RH}}{\rho_o}\right)^{1/3}$$  \hspace{1cm} (143)$$

In addition, we have

$$\rho_o \approx V(\phi = 0) = M^4$$  \hspace{1cm} (144)$$

and if one assumes that the time of reheating is close enough to the beginning of the radiation dominated era, the following is true

$$\rho_{RH} \approx \sigma T_{RH}^4 \approx T_{RH}^4$$  \hspace{1cm} (145)$$

where we used the fact that in our units is $\sigma \approx 1$. Substituting the last two equations into (140) we obtain

$$\lambda_{RH} = \left(\frac{M}{T_{RH}}\right)^{4/3} \lambda_f$$  \hspace{1cm} (146)$$

Now, based on the assumption that after the reheating and until today the expansion of a patch which now is our observable universe occurs adiabatically we have

$$dS = 0$$

and ,in addition, it is proved that a relation like

$$S \propto a^3 T^3$$  \hspace{1cm} (147)$$

must hold. Thus, we derive

$$a^3 T^3 = constant$$  \hspace{1cm} (148)$$

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where $T$ is the temperature of the universe, and by applying the above for today and reheating period and recalling that $\lambda \propto a$ it follows that

$$\lambda_{RH}^3 T_{RH}^3 = \lambda_o^3 T_o^3$$  \hspace{1cm} (149)$$

where the index $o$ denotes today. We know that the microwave background radiation, today, has a value at about $T_o \approx 2.73 K \approx 10^{-13} GeV$ and our observable universe is $\lambda_o = 10^4 Mpc$, so we have $(\lambda_o T_o)^3 \approx 10^{88}$, such that the last gives

$$\lambda_{RH} = \frac{10^{88/3}}{T_{RH}}$$  \hspace{1cm} (150)$$

and by substituting the last into (143) we obtain

$$\lambda_f = 10^{88/3} M^{-4/3} T_{RH}^{1/3}$$  \hspace{1cm} (151)$$

also we can rewrite equation (147) as

$$\lambda_{H.C} = \frac{m_{pl}}{M^2} \sqrt{3\Omega_{H.C}}$$  \hspace{1cm} (152)$$

and the above two equations together with (138) are combined to give

$$e^N = T_{RH} M^{-4/3} 10^{88/3} m_{pl}^{-1} (3\Omega_{H.C})^{-1/2}$$

and by taking the logarithm it follows that

$$N = 2 \ln M + \frac{1}{3} \ln T_{RH} - \frac{1}{2} \ln (3\Omega_{H.C}) + \frac{88}{3} \ln 10 - \ln m_{pl}$$  \hspace{1cm} (154)$$

susbituting $\Omega_{H.C}$ from (129) expressing $M$ and $T_{RH}$ in units of $10^{14} GeV$ and $10^{10} GeV$ respectively and using the fact that $m_{pl} = \sqrt{\frac{\hbar c}{G}} \approx 1,2 \cdot 10^{19} GeV$ we derive

$$N = 46 + \frac{2}{3} \ln M/10^{14} + \frac{1}{3} \ln \frac{T_{RH}}{10^{10}} - \frac{1}{2} \ln (\Omega_{H.C})$$  \hspace{1cm} (155)$$

where as usual $K = \pm 1$, and as has been mentioned the units for $M$ and $T_{RH}$ are $10^{14} GeV$ and $10^{10} GeV$ respectively. The last gives us the number of $e - folds$ contained between the time when a length, which today has the dimensions of our observable universe, crosses the horizon until the end of inflation. Now, in order to have an expression for the number of $e - folds$ contained between the time when a random wavelength crosses the horizon until the end of inflation, we start by the definition of number $N$ of $e - folds$. For a scale $\lambda$, by the definition we have

$$N_\lambda = \ln \left( \frac{\lambda}{\lambda_f} \right) = \ln \left( \frac{\lambda}{\lambda_o \lambda_f} \right) \Rightarrow$$
\[ N_\lambda = \ln \left( \frac{\lambda}{\lambda_o} \right) + N \Rightarrow \]
\[ N_\lambda = \ln \left( \frac{\lambda}{\lambda_o} \right) + 46 + \frac{2}{3} \ln \left( \frac{M}{10^{14}} \right) + \frac{1}{3} \ln \left( \frac{T_{RH}}{10^{10}} \right) - \frac{1}{2} \ln \left( \Omega_{H.C.} \right) \] (157)
8 Conclusions-Epilogue

In the present thesis we saw how the curvature influences the expansion of the universe during inflation. More specifically, at the beginning of inflation the curvature of the universe slows down the expansion and causes a linear increasing of the scale factor as a function of time. Though, at the ending of inflation the whole curvature has been "gone away" and one has the known exponential expansion of the scale factor. In addition, both for $K = -1$ and $K = +1$ we determined, for a given length, the time when it crosses the horizon, until the end of inflation and we had expressed this time as a function of the wavenumber of the length. From our results we can say that the proper time is slowly varying with the wavenumber as it grows logarithmically with the last. After that, we had computed the number of $e - folds$ remaining from inflation to end after a specific length has crossed the horizon. In this case we saw that when having a curved universe in the expression for the $e - folds$, a logarithmically term is added, which contains the wavenumber of the given length, in contrast to the flat case where this term vanishes.