Models of black hole disks with self-gravity and non-constant angular momentum

Author: Christos Vourellis

Supervisor: Prof. Nikolaos Stergioulas

A thesis submitted in fulfilment of the requirements for the degree of Master in Science in Computational Physics in the Department of Physics

October 2014
Models of black hole disks with self-gravity and non-constant angular momentum

by Christos Vourellis

We construct the first numerical models of general relativistic, self-gravitating accretion tori around black holes with a non-constant specific angular momentum distribution. The models are axisymmetric and stationary and the specific angular momentum increases outwards as a power law. This generalizes previous cases, where only constant specific angular momentum was considered or where the models were not self-gravitating. The non-constancy of specific angular momentum was shown to stabilize the disk against the axisymmetric runaway instability while self-gravity was neglected. We are thus particularly interested in models that are massive and form an inner cusp. As a first application, we study the existence of equilibrium models in a restricted region of the parameter space and show that self-gravitating massive tori lie on a different surface of equilibrium models, that features an overlap in mass.
Acknowledgements

I want to thank my advisor for giving me the change to work in such an advanced and competitive scientific field and for his constant encouragement and support throughout the whole project. I also want to thank the professors of the masters program for their efforts to teach me and my colleagues as much as they could . . .
# Contents

Abstract ii

Acknowledgements iii

Contents iv

List of Figures vi

List of Tables ix

1 Introduction 1

2 Theoretical Background 3

2.1 Tori in equilibrium 3

2.2 Non Constant specific angular momentum 8

2.3 Self-gravitating equilibrium disks 11

2.4 Torus properties 17

3 Numerical method and various tori cases. 18

3.1 Numerical framework 18

3.2 Detailed approach to the constructed models 21

3.2.1 Unstable models 21

3.2.1.1 Totally unstable configurations 21

3.2.1.2 Configurations between the totally unstable limit and the cusp 26

3.2.2 Models at the cusp limit 31

3.2.3 Beyond the cusp limit 36

3.2.4 Models at the $W_{\text{peak}} = 0$ limit 40

3.2.5 Approaching the upper limit 44

3.2.6 The upper limit 49

4 The parameter space of equilibrium models 54

4.1 The subspace for the specific angular momentum parameters $K_{\text{EOS}}^{\text{AJS}} = 0.2$ and $r_{\text{out}}^{\text{AJS}} = 39$ 54

4.2 The subspace for the specific angular momentum parameters $K_{\text{EOS}}^{\text{AJS}} = 0.2$ and $r_{\text{out}}^{\text{AJS}} = 60$ 61
4.3 The subspace of the specific angular momentum parameters for $K_{E}\Sigma = 0.3$ and $r_{out}^{AJS} = 39$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 68

4.4 The subspace of the specific angular momentum parameters for $K_{E}\Sigma = 0.3$ and $r_{out}^{AJS} = 60$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 74

5 Conclusions and comments 80
List of Figures

3.1 The effective gravitational potential for an unstable torus. ............. 22
3.2 The isopotential contours of an unstable torus. ......................... 23
3.3 The four metric functions of an unstable torus in comparison with the ones in the AJS torus. ........................................... 25
3.4 The effective potential of a torus between the totally unstable and cusp limits. ............................................................... 26
3.5 The isopotential contours of a torus between the totally unstable and cusp limits. ............................................................... 27
3.6 The four metric functions for a torus between the totally unstable and the cusp limits in comparison with the ones in the AJS torus. ........ 29
3.7 The effective potential of a torus filling exactly its Roche lobe. ........ 31
3.8 The isopotential contours of a torus filling exactly its Roche lobe. .... 32
3.9 The four metric functions for a torus in the cusp limit in comparison with the ones in the AJS torus. ............................................. 35
3.10 The effective potential of a torus beyond the cusp limit. .................. 36
3.11 The isopotential contours of a torus beyond the cusp limit. ............. 37
3.12 The four metric functions for a torus beyond the cusp limit in comparison with the ones in the AJS torus. ..................................... 39
3.13 The effective potential of a torus in the $W_{\text{peak}} = 0$ limit. .......... 40
3.14 The isopotential contours of a torus in the $W_{\text{peak}} = 0$ limit. ........ 41
3.15 The four metric functions for a torus at the $W_{\text{peak}} = 0$ limit. ........ 43
3.16 The effective potential of a torus approaching the upper limit. ........ 44
3.17 The isopotential contours of a torus approaching the upper limit. .... 45
3.18 The four metric functions for a torus approaching the upper limit in comparison with the ones in the AJS torus. ........................ 48
3.19 The effective potential of the torus just before the location of maximum density reaches the outer edge of the torus. ......................... 49
3.20 The isopotential contours of the torus just before the location of maximum density reaches the outer edge of the torus. ..................... 50
3.21 The four metric functions for a torus just before the location of maximum density reaches the outer edge of the torus. ..................... 53

4.1 The subspace of the specific angular momentum parameters for $K_{\text{EOS}}^\text{AJS} = 0.2$ and $r_{\text{out}}^\text{AJS} = 39$ models. (a) for the initial trial values of the constant $k$, (b) for the rescaled values. ................................................. 56
4.2 The effective potential for the cases presented in Figures 4.1 for tori with $K_{\text{EOS}}^\text{AJS} = 0.2$ and $r_{\text{out}}^\text{AJS} = 39$. ................................. 57
4.3 The dependence of the torus mass from the (input and rescaled) specific angular momentum parameters for $K_{\text{EOS}}^{\text{AJJS}} = 0.2$ and $r_{\text{out}}^{\text{AJJS}} = 39$. (a) is for the initial trial values of the constant $k$, while (b) is for the rescaled values. 58

4.4 The contour plot of the three-dimensional data. The torus mass as a function of the two specific angular momentum parameters for $K_{\text{EOS}}^{\text{AJJS}} = 0.2$ and $r_{\text{out}}^{\text{AJJS}} = 39$ models. 59

4.5 The parametric plot showing the dependence of the specific angular momentum constant from the torus mass and the critical torus mass for different values of slope $q$, for $K_{\text{EOS}}^{\text{AJJS}} = 0.2$ and $r_{\text{out}}^{\text{AJJS}} = 39$ models. 60

4.6 The subspace of models for the specific angular momentum parameters for $K_{\text{EOS}}^{\text{AJJS}} = 0.2$ and $r_{\text{out}}^{\text{AJJS}} = 60$. (a) is for the initial trial values of the constant $k$, while (b) is for the rescaled values. 63

4.7 The effective potential for the cases presented in Figures 4.6 for tori with $K_{\text{EOS}}^{\text{AJJS}} = 0.2$ and $r_{\text{out}}^{\text{AJJS}} = 60$. 64

4.8 The dependence of the torus mass on the (input and rescaled) specific angular momentum parameters for $K_{\text{EOS}}^{\text{AJJS}} = 0.2$ and $r_{\text{out}}^{\text{AJJS}} = 60$. (a) is for the initial trial values of the constant $k$, while (b) is for the rescaled values. 65

4.9 The torus mass as a function of the two specific angular momentum parameters for $K_{\text{EOS}}^{\text{AJJS}} = 0.2$ and $r_{\text{out}}^{\text{AJJS}} = 60$ models. 66

4.10 Parametric plot showing the dependence of the specific angular momentum constant from the torus mass and the critical torus mass for different values of exponent $q$, for $K_{\text{EOS}}^{\text{AJJS}} = 0.2$ and $r_{\text{out}}^{\text{AJJS}} = 60$. 67

4.11 The subspace of the specific angular momentum parameters for $K_{\text{EOS}}^{\text{AJJS}} = 0.3$ and $r_{\text{out}}^{\text{AJJS}} = 39$ models. (a) is for the initial trial values of the constant $k$, while (b) is for the rescaled values. 69

4.12 The effective potential for the cases presented in Figures 4.11 for tori with $K_{\text{EOS}}^{\text{AJJS}} = 0.3$ and $r_{\text{out}}^{\text{AJJS}} = 39$. 70

4.13 The dependence of the torus mass from the (input and rescaled) specific angular momentum parameters for $K_{\text{EOS}}^{\text{AJJS}} = 0.3$ and $r_{\text{out}}^{\text{AJJS}} = 39$. (a) is for the initial trial values of the constant $k$, while (b) is for the rescaled values. 71

4.14 The contour plot of the three-dimensional data for $K_{\text{EOS}}^{\text{AJJS}} = 0.3$ and $r_{\text{out}}^{\text{AJJS}} = 39$. The torus mass is shown as a function of the two specific angular momentum parameters. 72

4.15 Parametric plot showing the dependence of the specific angular momentum constant from the torus mass and the critical torus mass for different values of exponent $q$, for $K_{\text{EOS}}^{\text{AJJS}} = 0.3$ and $r_{\text{out}}^{\text{AJJS}} = 39$. 73

4.16 The subspace of the specific angular momentum parameters for $K_{\text{EOS}}^{\text{AJJS}} = 0.3$ and $r_{\text{out}}^{\text{AJJS}} = 60$. (a) is for the initial trial values of the constant $k$, while (b) is for the rescaled values. 75

4.17 The effective potential for the cases presented in Figures 4.16 for tori with $K_{\text{EOS}}^{\text{AJJS}} = 0.3$ and $r_{\text{out}}^{\text{AJJS}} = 60$. 76

4.18 The dependence of the torus mass on the (input and rescaled) specific angular momentum parameters for $K_{\text{EOS}}^{\text{AJJS}} = 0.3$ and $r_{\text{out}}^{\text{AJJS}} = 60$. (a) is for the initial trial values of the constant $k$, while (4.18b) is for the rescaled values. 77

4.19 Contour plot of the three-dimensional data showing in figure 4.3. The color scale corresponds to the torus mass. 78
4.20 Parametric plot showing the dependence of the specific angular momentum constant from the torus mass and the critical torus mass $M_{ct}$ for overlap, for different values of exponent $q$, for $K_{EOS}^{\text{AJS}} = 0.3$ and $r_{out}^{\text{AJS}} = 60$. 79

5.1 The changes in the outer radius for the higher values of the torus mass. 82
5.2 The changes in the equation of state constant for the higher values of the torus mass. 84
# List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Detailed information about the representative model of an unstable torus.</td>
<td>22</td>
</tr>
<tr>
<td>3.2</td>
<td>Detailed information about the torus between the totally unstable and the cusp limits.</td>
<td>28</td>
</tr>
<tr>
<td>3.3</td>
<td>Detailed information about the torus in the cusp limit.</td>
<td>33</td>
</tr>
<tr>
<td>3.4</td>
<td>Detailed information about the torus beyond the cusp limit.</td>
<td>37</td>
</tr>
<tr>
<td>3.5</td>
<td>Detailed information about the torus at the $W_{\text{peak}} = 0$ limit.</td>
<td>41</td>
</tr>
<tr>
<td>3.6</td>
<td>Detailed information about the torus approaching the upper limit.</td>
<td>46</td>
</tr>
<tr>
<td>3.7</td>
<td>Detailed information about the torus just before the location of maximum density reaches the outer edge of the torus.</td>
<td>51</td>
</tr>
<tr>
<td>5.1</td>
<td>The minimum, mean and maximum value of the torus mass which causes the overlap in mass.</td>
<td>80</td>
</tr>
</tbody>
</table>
To my parents...
Chapter 1

Introduction

Black holes are one of the final products of stellar evolution. Stellar-mass black holes can be created either by the collapse of the core of a massive dying star by the accretion of matter onto a neutron star, or through the merge of two neutron stars.

An accretion disk is most likely to be formed in the environment of a binary star system, where the more massive of the two stars evolves faster than the other, which results in its death leaving behind a stellar remnant (white dwarf, neutron star or black hole). These three stellar remnants share the characteristic of a gravitational field so strong, that when the companion star enters the later stages of its evolution and its matter exceeds its Roche lobe, some of the gas from it will start flowing onto the primary star. The conservation of the angular momentum prevents the matter to flow straight onto the remnant star, which causes a disk to be formed.

Apart from their involvement in the evolution of binary star systems, accretion disks are known to exist, on a much larger scale, in the center of galaxies around supermassive black holes. The gradient of the gravitational field is strong enough to raise the temperature of the gas in the disk and cause it to emit X-ray radiation. Radiation observed from active galactic nuclei and quasars, comes from physical processes involving supermassive black holes and their accretion disks.

The first approach in constructing equilibrium models of tori around black holes was presented in 1978 by Abramowicz, Jaroszynski and Sikora, who studied tori in a fixed spacetime (ignoring the disks self-gravity), the so called AJS disks [1]. In these disks the spacetime is determined only by the properties of an isolated black hole. In 1986, Hachisu [2] developed a self-consistent field method using Newtonian gravity to create self-gravitating stars and disks in axisymmetry [3, 4], following the theoretical work of Butterworth and Ipser [5] and in 1989 Komatsu, Eriguchi and Hachisu applied the
original idea to rotating relativistic stars [6]. In 1994 Nishida and Eriguchi developed further the computational method by introducing new boundary conditions on the event horizon and thus presented the first self-gravitating disks with constant specific angular momentum. In 2011 Stergioulas [7, 8] adopted the compactified grid present by Cook, Shapiro and Teukolsky [9] and extended Nishida and Eriguchi’s method [10] focusing on self-gravitating tori with constant specific angular momentum, which fill exactly their Roche lobe.

On the other hand, Daigne and Font [11] presented disks with non-constant specific angular momentum, that changes with the radial distance following a power law, but still ignoring self-gravity.

In the present thesis, we implement the non-constant specific angular momentum distribution in the numerical code of Stergioulas [7] and present a first study of equilibrium models in a restricted region of the parameter space.

The thesis is organized as follows: In Chapter 2 we present the definitions and the mathematical background of the theory of self-gravitating tori. In Chapter 3 we describe briefly the numerical algorithm and present some of the different types of disks models created with the computational method. In Chapter 4 we explore a region of the parameter space of equilibrium models. A discussion of the results closes the thesis in Chapter 5.

We use a spacetime signature of $(-, +, +, +)$ and set $c = G = 1$. 
Chapter 2

Theoretical Background

In this Chapter, we summarize the main theory based on Stergioulas [7] and Friedmann and Stergioulas [12].

2.1 Tori in equilibrium

The line element that describes a stationary and axisymmetric spacetime can be written in the form

\[ ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2. \]  

(2.1)

Since the torus is stationary and we do not assume meridional currents, the fluid has a 4-velocity

\[ u^\alpha = \left( u^t, 0, 0, u^\phi \right) \]  

(2.2)

and a specific enthalpy

\[ h = \frac{(\epsilon + p)}{\rho}, \]  

(2.3)

where \( \epsilon \) is the energy density, \( p \) is the pressure and \( \rho \) is the rest mass density of the fluid.

The angular velocity of the torus (as defined by an observer at infinity) is

\[ \Omega = \frac{u^\phi}{u^t}. \]  

(2.4)
The existence of the Killing vectors $t^\alpha$ and $\phi^\alpha$ that describe the time and azimuthal symmetry result in the conservation of the energy per unit rest mass, $E = -hu_t$ and in the conservation of the angular momentum per unit rest mass, $j = hu_\phi$, respectively.

The specific angular momentum $l$ is also conserved, since it results from the ratio

$$ l := \frac{j}{E} = \frac{hu_\phi}{-hu_t} = -\frac{u_\phi}{u_t} . \quad (2.5) $$

The angular momentum per unit mass and the energy per unit mass can be written as a linear combination of the metric tensor components

$$ j = hu_\phi = \frac{1}{2} h \left( g_{t\phi} u^t + g_{\phi\phi} u^\phi \right) , $$

$$ E = -hu_t = -\frac{1}{2} h \left( g_{t\phi} u^\phi + g_{tt} u^t \right) . $$

So, the specific angular momentum is

$$ l = -\frac{g_{t\phi} u^t + g_{\phi\phi} u^\phi}{g_{t\phi} u^\phi + g_{tt} u^t} = -\frac{g_{t\phi}}{g_{tt}} + \frac{\Omega g_{\phi\phi}}{g_{tt}} , \quad (2.6) $$

and solving for $\Omega$:

$$ \Omega = -\frac{g_{t\phi} + lg_{tt}}{g_{\phi\phi} + lg_{t\phi}} . \quad (2.7) $$

Normalizing the 4-velocity

$$ u^\alpha u_\alpha = -1 , $$

$$ \Rightarrow g^{\alpha\beta} u_\beta u_\alpha = -1 , $$

$$ \Rightarrow g^{tt}(u_t)^2 + 2g^{t\phi} u_t u_\phi + g^{\phi\phi}(u_\phi)^2 = -1 , \text{[since } u_r = u_\theta = 0\text{]} , $$

$$ \Rightarrow g^{tt} + 2g^{t\phi} \frac{u_t u_\phi}{(u_t)^2} + g^{\phi\phi} \left( \frac{u_\phi}{u_t} \right)^2 = -\frac{1}{(u_t)^2} $$

$$ \Rightarrow g^{tt} - 2g^{t\phi} l + g^{\phi\phi} l^2 = -\frac{1}{(u_t)^2} , \text{[since } l = -\frac{u_\phi}{u_t}\text{]} $$

$$ \Rightarrow u_t = -\left( -g^{tt} + 2lg^{t\phi} - l^2 g^{\phi\phi} \right)^{-1/2}. \quad (2.8) $$
By calculating the inverse matrix of the metric tensor $g^{\alpha\beta}$ we find the components of the inverse metric

\[
\begin{align*}
g^{tt} &= \frac{1}{g_{tt} - (g_{t\phi})^2/g_{\phi\phi}}, \quad (2.9a) \\
g^{rr} &= \frac{1}{g_{rr}}, \quad (2.9b) \\
g^{\theta\theta} &= \frac{1}{g_{\theta\theta}}, \quad (2.9c) \\
g^{\phi\phi} &= \frac{1}{g_{\phi\phi} - (g_{t\phi})^2/g_{tt}}, \quad (2.9d) \\
g^{t\phi} &= \frac{g_{t\phi}}{g_{tt}g_{\phi\phi} - (g_{t\phi})^2}, \quad (2.9e)
\end{align*}
\]

and we use them to write $u_t$ as a function of the covariant components of the metric tensor

\[
u_t = -\left[\frac{g_{\phi\phi} + 2lg_{t\phi} + l^2g_{tt}}{g_{tt}g_{\phi\phi} - (g_{t\phi})^2}\right]^{-1/2}.
\]

(2.10)

We repeat the same normalization as before, but now we use the covariant metric $g_{\alpha\beta}$ and obtain the contravariant form $u^t$

\[
u^t = \left(-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi}\right)^{1/2}.
\]

(2.11)

Again, from the normalization

\[
\begin{align*}
u^\alpha \nu_\alpha &= -1, \\
\Rightarrow \nu^t \nu_t + \nu^\phi \nu_\phi &= -1, \\
\Rightarrow \nu^t \nu_t &= 1 - \nu^\phi \nu_\phi, \\
\Rightarrow \nu^t \nu_t &= 1 - \Omega \nu_t(-l \nu_t), \\
\Rightarrow \nu^t \nu_t(1 - \Omega l) &= -1, \\
\Rightarrow \nu^t \nu_t &= -\frac{1}{1-\Omega l},
\end{align*}
\]

(2.12)

and because of (2.5)
\[ u^t u_\phi = \frac{l}{1 - \Omega l}. \] (2.13)

The equation of hydrostationary equilibrium is [12]

\[ \frac{\nabla p}{\epsilon + p} = \nabla \ln u^t - u^t u_\phi \nabla \Omega, \] (2.14)

and using (2.13)

\[ \frac{\nabla p}{\epsilon + p} = \nabla \ln u^t - \frac{\nabla \Omega}{1 - \Omega l}. \] (2.15)

From (2.15) and (2.12) we obtain

\[
\frac{\nabla p}{\epsilon + p} = \nabla \ln \left[ - \frac{1}{u_t (1 - \Omega l)} \right] - \frac{\nabla \Omega}{1 - \Omega l},
\] (2.16a)

We also know that

\[- \nabla \ln (1 - \Omega l) = - \frac{1}{1 - \Omega l} \nabla (1 - \Omega l), \] (2.16b)

and

\[ l \nabla \Omega = \nabla (\Omega l) - \Omega \nabla l \] (2.16c)

Combining (2.16a), (2.16b) and (2.16c)
\[
\frac{\nabla p}{\epsilon + p} = -\nabla \ln(-u_t) - \nabla \ln(1 - \Omega l) - \frac{l \nabla \Omega}{1 - \Omega l} \\
= -\nabla \ln(-u_t) - \frac{1}{1 - \Omega l} \nabla (1 - \Omega l) - \frac{l \nabla \Omega}{1 - \Omega l} \\
= -\nabla \ln(-u_t) - \frac{\nabla (1 - \Omega l) + l \nabla \Omega}{1 - \Omega l} \\
= -\nabla \ln(-u_t) - \frac{\nabla (1 - \Omega l) - \nabla (1 - \Omega l) - \Omega \nabla l}{1 - \Omega l},
\]

and we obtain the following form of the equation of stationary equilibrium

\[
\frac{\nabla p}{\epsilon + p} = -\nabla \ln(-u_t) + \frac{\Omega \nabla l}{1 - \Omega l}.
\tag{2.17}
\]

If we consider a barotropic equation of state for the fluid in the form \( p = p(\rho) \) we can define the log-enthalpy \( H(p) \)

\[
H(p) = \int_0^p \frac{dp'}{\epsilon(p') + p'},
\tag{2.18}
\]

where \( \frac{dH}{dp} = \frac{d}{dp} \int_0^p \frac{dp'}{\epsilon(p') + p'} = \frac{1}{\epsilon + p} \). Then, \( \nabla H = \nabla p \frac{dH}{dp} = \frac{\Sigma p}{\epsilon + p} \). The right side of (2.17) can be written as the gradient of a potential only, if \( \Omega = \Omega(l) \). Then,

\[
\nabla H = \nabla \left[ -\ln(-u_t) + \int_1^l \frac{\Omega}{1 - \Omega l} dl \right] := -\nabla W,
\tag{2.19}
\]

where

\[
W = \ln(-u_t) - \int_1^l \frac{\Omega}{1 - \Omega l} dl + \text{const.}
\tag{2.20}
\]

is the effective potential and \( -\nabla W \) the effective gravity. Since \( \nabla H = -\nabla W \), the surfaces of constant enthalpy and the surfaces of constant effective potential are the same. From (2.19) we obtain the first integral of the equations of motion

\[
H + \ln(-u_t) - \int_1^l \frac{\Omega}{1 - \Omega l} dl = \text{const.}
\tag{2.21}
\]

Since we assumed that the entropy of the fluid is constant throughout the torus, the log-enthalpy becomes
\[ H = \ln \left( \frac{h}{h_{\text{min}}} \right), \quad (2.22) \]

where \( h_{\text{min}} = 1 \) is the specific enthalpy at the surface of the torus (zero pressure) where \( H = 0 \).

### 2.2 Non Constant specific angular momentum

We assume that the specific angular momentum in the equatorial plane of the torus changes with the radial distance \( r \) following a power law

\[ l_{\text{eq}} = k R^q, \quad (2.23) \]

where \( R \) is the radial distance in the equatorial plane in Schwarzschild coordinates and \( k \) a constant value. The above equation is transformed into isotropic Schwarzschild coordinates, using

\[ R = \frac{M^2}{4r} + M + r, \quad (2.24) \]

where \( r \) is the isotropic radial coordinate \([7]\). Assuming that the value of the specific angular momentum in the equatorial plane at radius \( r_0 \) is \( l_{\text{eq}}(r_0) \), the equation that describes the surface with constant specific angular momentum is

\[
0 = Z(r, \theta; r_0) = \left[ g_{tt} \tilde{g}_{\phi\phi}(r_0) - g_{t\phi} \tilde{g}_{tt}(r_0) \right] l_{\text{eq}}^2(r_0) \\
+ \left[ g_{tt} \tilde{g}_{\phi\phi}(r_0) - g_{\phi\phi} \tilde{g}_{tt}(r_0) \right] l_{\text{eq}}(r_0) \\
+ \left[ g_{t\phi} \tilde{g}_{\phi\phi}(r_0) - g_{\phi\phi} \tilde{g}_{t\phi}(r_0) \right]. \quad (2.25)
\]

and the respective surfaces are called \textit{von Zeipel cylinders} \([11]\). For an AJS torus the components of the metric tensor are known functions of the coordinates, \( g_{\alpha\beta} = g_{\alpha\beta}(r, \theta) \) and the above equation analytically represents the surfaces of constant specific angular momentum. In a self-gravitating torus though, the metric tensor cannot be expressed analytically and the above equation for the von Zeipel’s cylinders cannot be used.

As a first approximation, we assume that (2.23) holds also for any other angle away from the equatorial plane. For tori that do not reach high angles, this is an acceptable approximation for studying the effect of (2.23) in contrast to assuming that \( l = \text{const} \). A more elaborate approach would involve solving (2.6) in addition to (2.7) self-consistently during the numerical iteration process.
Chapter 2. Theoretical Background

The effective potential at any given point is defined up to a constant according to (2.20). We assume the spacetime is asymptotically flat. Then, we can write the effective potential as

$$ W_\infty = \ln(-u_{t,\infty}) - \int^{l_\infty} \frac{\Omega}{1 - \Omega l} dl + \text{const.} \quad (2.26) $$

where $-u_{t,\infty}$ and $l_\infty$ are the respective values of the time component of the 4-velocity and the specific angular momentum at infinity. From (2.20) and (2.26)

$$ W - W_\infty = \ln(-u_t) - \int^{l} \frac{\Omega}{1 - \Omega l} dl - \ln(-u_{t,\infty}) + \int^{l_\infty} \frac{\Omega}{1 - \Omega l} dl $$

$$ = \ln\left(\frac{-u_t}{-u_{t,\infty}}\right) + \int_{l}^{l_\infty} \frac{\Omega}{1 - \Omega l} dl + \int^{l_\infty} \frac{\Omega}{1 - \Omega l} dl $$

$$ = \ln\left(\frac{-u_t}{-u_{t,\infty}}\right) + \int^{l_\infty} \frac{\Omega}{1 - \Omega l} dl, \quad (2.27) $$

and since $W_\infty = 0$ and $u_{t,\infty} = -1$

$$ W = \ln(-u_t) + \int^{l_\infty} \frac{\Omega}{1 - \Omega l} dl. \quad (2.28) $$

Following the same calculations as before, the difference $W - W_{\text{out}}$ (where out indicates the outer edge of the torus) is

$$ W - W_{\text{out}} = \ln\left(\frac{-u_t}{-u_{t,\text{out}}}\right) + \int_{l}^{l_{\text{out}}} \frac{\Omega}{1 - \Omega l} dl, \quad (2.29) $$

From $\nabla H = -\nabla W$,

$$ H - H_{\text{out}} = \ln\left(\frac{-u_{t,\text{out}}}{-u_t}\right) - \int_{l}^{l_{\text{out}}} \frac{\Omega}{1 - \Omega l} dl, \quad (2.30) $$

but as we have already mentioned, the log-enthalpy is equal to zero at the surface of the torus which means that $H_{\text{out}} = 0$.

Thus, the log-enthalpy

$$ H = W_{\text{out}} - W. \quad (2.31) $$

Assuming a homentropic, polytropic equation of state in the form of
\[ p = K \rho^\Gamma, \quad (2.32) \]
\[ \epsilon = \rho + \frac{p}{\Gamma - 1}, \quad (2.33) \]

where \( K \) is the polytropic constant, \( \Gamma = 1 + 1/N \) the polytropic exponent and \( N \) the polytropic index.

From 2.22, 2.32 and 2.33

\[
H = \ln\left( \frac{h}{h_{\text{min}}} \right) = \ln h = \ln \left( \frac{\epsilon + p}{\rho} \right) = \ln \left[ \rho + \frac{p}{(\Gamma - 1) + p} \right] = \ln \left[ \rho + \frac{(K \rho^\Gamma/(\Gamma - 1)) + K \rho^\Gamma}{\rho} \right] = \ln \left[ 1 + \frac{K \rho^\Gamma - 1}{(\Gamma - 1) + 1} \right] = \ln \left( 1 + \frac{\Gamma}{\Gamma - 1} K \rho^\Gamma - 1 \right), \quad (2.34)
\]

By solving (2.34) we obtain the distribution of the density in the disk

\[
H = \ln \left( 1 + \frac{\Gamma}{\Gamma - 1} K \rho^\Gamma - 1 \right), \\
\rho = \left[ \frac{\Gamma - 1}{K \Gamma} (e^H - 1) \right]^{\frac{1}{\Gamma - 1}}, \quad (2.35)
\]

and the pressure of the disk

\[
p = \left[ \frac{\Gamma - 1}{K^{1/\Gamma} \Gamma} (e^H - 1) \right]^{\frac{\Gamma}{\Gamma - 1}}. \quad (2.36)
\]
2.3 Self-gravitating equilibrium disks

For the self-gravitating tori we will use a quasi-isotropic coordinate system, where the line element is given as

\[ ds^2 = -e^{2\nu} dt^2 + e^{2\alpha} dr^2 + r^2 d\theta^2 + e^{2(\gamma-\nu)} r^2 \sin^2 \theta (d\phi - \omega dt)^2, \]

(2.37)

where \( \nu, \gamma, \alpha, \omega \) are functions of the coordinates \( r \) and \( \theta \). Thus, the components of the metric tensor in (2.1) become

\[
\begin{align*}
    g_{tt} &= -e^{2\nu} + \omega^2 e^{2(\gamma-\nu)} r^2 \sin^2 \theta, \\
    g_{rr} &= e^{2\alpha}, \\
    g_{\theta\theta} &= r^2 e^{2\alpha}, \\
    g_{\phi\phi} &= e^{2(\gamma-\nu)} r^2 \sin^2 \theta, \\
    g_{t\phi} &= -\omega e^{2(\gamma-\nu)} r^2 \sin^2 \theta.
\end{align*}
\]

(2.38a) - (2.38e)

We define

\[
B := e^\gamma, \quad \lambda := e^\nu,
\]

(2.39)

because the initial metric functions become divergent when we approach the event horizon. With these new metric functions the event horizon satisfies the following boundary conditions

\[
\begin{align*}
    B_h &= 0, \\
    \lambda_h &= 0, \\
    \omega &= \omega_h,
\end{align*}
\]

(2.40) - (2.42)

where \( \omega_h \) is the angular velocity of the horizon. Instead of the two aforementioned coordinates \( r \) and \( \theta \), we use the compactified coordinate \( s \)

\[
r := r e^{\frac{s}{1-s}},
\]

(2.43)
Chapter 2. Theoretical Background

where \( r_e \) is the outer (in our case \( r_e = r_{out} \)) radius of the torus and the coordinate

\[
\mu = \cos \theta, \tag{2.44}
\]

instead of the polar angle \( \theta \). With these transformations we went from a domain where \( r \in [0, \infty) \) and \( \theta \in [0, \pi/2] \) to a more compactified and finite domain where \( s \in [0, 1] \) and \( \mu \in [0, 1] \).

From (2.11) and (2.38) in the quasi-isotropic spacetime the 4-velocity becomes

\[
\begin{align*}
\mathbf{u}^t &= \left[ -g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi} \right]^{1/2} \\
&= \left[ e^{2\nu} - \omega^2 e^{2(\gamma-\nu)} r^2 \sin^2 \theta + 2\Omega \omega e^{2(\gamma-\nu)} r^2 \sin^2 \theta - \Omega^2 e^{2(\gamma-\nu)} r^2 \sin^2 \theta \right]^{1/2} \\
&= \left[ e^{2\nu} - e^{2(\gamma-\nu)} r^2 \sin^2 \theta \left( \omega^2 - 2\Omega \omega + \Omega^2 \right) \right]^{-1/2} \\
&= \left[ \frac{e^{2\nu} e^{-2\nu} e^{-2\nu} r^2 \sin^2 \theta \left( \omega - \Omega \right)^2}{e^{-2\nu}} \right]^{-1/2} \\
&= \left[ 1 - e^{2\gamma} e^{-4\nu} r^2 \sin^2 \theta \left( \omega - \Omega \right)^2 \right]^{-1/2} \\
&= e^{-\nu} \\
&= \sqrt{1 - e^{2\gamma} e^{-4\nu} r^2 \sin^2 \theta \left( \omega - \Omega \right)^2} \\
&= e^{-\nu} \sqrt{1 - B^2 e^{-4\nu} r^2 \sin^2 \theta \left( \omega - \Omega \right)^2} \\
&= \frac{e^{-\nu}}{\sqrt{1 - v^2}}, \tag{2.45}
\end{align*}
\]

where

\[
v = (\Omega - \omega) B e^{-2\nu} r \sin \theta, \tag{2.46}
\]

is the proper velocity as it is measured by a zero specific angular momentum observer.

The general form of the Einstein field equations is

\[
G_{\alpha\beta} := R_{\alpha\beta} + \frac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{2.47}
\]
where $G_{\alpha\beta}, R_{\alpha\beta}, R, T_{\alpha\beta}$ are the Einstein tensor, the Ricci tensor, the scalar Ricci curvature and the energy-momentum tensor, respectively. We assume that the torus forms from a perfect fluid with an energy-momentum tensor given by

$$T_{\alpha\beta} = (\epsilon + p) u_\alpha u_\beta + pg_{\alpha\beta}. \quad (2.48)$$

The three components of the Einstein field equations are reduced to equations of elliptic type for the metric functions $\lambda, B$ and $\omega$.

$$\nabla^2 \lambda = S_\lambda(s, \mu), \quad (2.49)$$

$$\left( \nabla^2 + \frac{(1 - s)^3}{r_e^2 s} \frac{\partial}{\partial s} - \frac{(1 - s)^2}{r_e^2 s^2} \frac{\partial}{\partial \mu} \right) B = S_B(s, \mu), \quad (2.50)$$

$$\left( \nabla^2 + \frac{2(1 - s)^3}{r_e^2 s} \frac{\partial}{\partial s} - \frac{2(1 - s)^2}{r_e^2 s^2} \frac{\partial}{\partial \mu} \right) \omega = S_\omega(s, \mu), \quad (2.51)$$

where

$$S_\lambda := 4\pi \lambda e^{2\alpha} \left[ (\epsilon + p) \frac{1 + v^2}{1 - v^2} + 2p \right]$$

$$+ \frac{1}{2}(-\mu^2)B^2 \lambda^{-3} \left[ s^2 (1 - s)^2 (\omega, s)^2 + (1 - \mu^2)(\omega, \mu)^2 \right]$$

$$- \frac{(1 - s)^2}{r_e^2} \left[ (1 - s)^2 (\gamma, s - \nu, s) \lambda, s - \frac{1 - \mu^2}{s^2}(\gamma, \mu - \nu, \mu) \lambda, \mu \right], \quad (2.52)$$

$$S_B := 16\pi p B e^{2\alpha}, \quad (2.53)$$

$$S_\omega := \frac{(1 - s)^2}{r_e^2} \left[ (1 - s)^2 (4\gamma, s - 3\nu, s) \omega, s + \frac{1 - \mu^2}{s^2}(4\gamma, \mu - 3\nu, \mu) \omega, \mu \right]$$

$$- 16\pi e^{2\alpha}(\epsilon + p) \frac{\Omega - \omega}{1 - v^2}, \quad (2.54)$$

are the source terms of the equations. The partial differentiation of the various terms is denoted by $,s$ and $,\mu$ for each coordinate, respectively.

Using appropriate Green’s functions the differential equations are inverted as
\[
\lambda = 1 - \frac{h_0}{r} - \sum_{n=0}^{\infty} P_{2n}(\mu)
\]

\[
\int_{h_0}^{\infty} dr' \int_{0}^{1} d\mu' r'^2 f_n^{(2)}(r, r') P_{2n}(\mu') S_\lambda(r', \mu'),
\]

\[
B = 1 - \frac{h_0}{r} - \frac{2\pi}{r \sin \theta} \sum_{n=1}^{\infty} \frac{\sin(2n - 1)\theta}{2n - 1}
\]

\[
\int_{h_0}^{\infty} dr' \int_{0}^{1} d\mu' r'^2 j_{2n-1}(r, r') \sin(2n - 1)\theta S_B(r', \mu'),
\]

\[
\omega = \frac{\omega h_0^3}{r^3} - \frac{1}{r \sin \theta} \sum_{n=1}^{\infty} \frac{P_{2n-1}(\mu)}{2n(2n - 1)}
\]

\[
\int_{h_0}^{\infty} dr' \int_{0}^{1} d\mu' r'^3 \sin \theta' j_{2n-1}(r, r') P_{2n-1}^1(\mu') S_\omega(r', \mu'),
\]

where

\[
f_n^{(1)}(r, r') := \begin{cases} \left( \frac{r'}{r} \right)^n - \frac{h_0^{2n}}{(rr')^n}, & r' \leq 1, \\ \left( \frac{r'}{r} \right)^n - \frac{h_0^{2n}}{(rr')^n}, & r' > 1, \end{cases}
\]

\[
f_n^{(2)}(r, r') := \begin{cases} \frac{1}{r} \left( \frac{r'}{r} \right)^n - \frac{h_0^{2n+1}}{(rr')^{n+1}}, & r' \leq 1, \\ \frac{1}{r} \left( \frac{r'}{r} \right)^n - \frac{h_0^{2n+1}}{(rr')^{n+1}}, & r' > 1, \end{cases}
\]

and

\[
h_0 = r_e \frac{s_0}{1 - s_0}
\]

at the horizon.

The metric function \( \lambda \) is

\[
\lambda(s, \mu) = 1 - \frac{s_0(1 - s)}{s(1 - s_0)} - \sum_{n=0}^{\infty} P_{2n}(\mu) D_n^{(2)}(s),
\]

where
Chapter 2. Theoretical Background

The metric function $B$ is

$$B(s, \mu) = 1 - \left[ \frac{s_0(1 - s)}{s(1 - s_0)} \right]^2 - 2\pi r^2 \sum_{n=1}^{\infty} \Pi_{2n-1}^1(\mu) D_n^{B(2)}(s), \tag{2.62a}$$

where

$$D_n^{B(2)}(s) := \int_{s_0}^{1} ds' f_n^{B(1)}(s, s') D_n^{B(1)}(s'), \tag{2.62b}$$

$$D_n^{B(1)}(s) := \int_{0}^{1} d\mu' \sin(2n - 1)\theta' S_B(s', \mu'), \tag{2.62c}$$

$$f_n^{B(1)}(s, s') := \frac{s^2}{(1 - s')^2} \frac{1 - s}{s} f_n^{B(1)}(s, s'), \tag{2.62d}$$

$$\Pi_{2n-1}^1(\theta) := \frac{\sin(n\theta)}{n \sin \theta}. \tag{2.62e}$$

The metric function $\omega$ is

$$\omega(s, \mu) = \left[ \frac{s_0(1 - s)}{s(1 - s_0)} \right]^3 \omega_h - r_c^2 \sum_{n=1}^{\infty} \Pi_n^2(\mu) D_n^{\omega(2)}(s), \tag{2.63a}$$

where
\[
D_n^{\omega(2)}(s) := \int_{s_0}^1 ds' \tilde{f}_{2n-1}^{(3)}(s, s') D_n^{\omega(1)}(s'), \\
D_n^{\omega(1)}(s') := \int_0^1 d\mu' \sqrt{1 - \mu'^2} P_{2n-1}^{1}(\mu') S_\omega(s', \mu'), \\
\tilde{f}_{2n-1}^{(3)}(s, s') := \frac{s^2(1-s)}{s(1-s')^2} f_{n}^{(2)}(s, s'), \\
\Pi_n^2(\mu) := \frac{P_{2n-1}^{1}(\mu)}{2n(2n-1)\sqrt{1 - \mu^2}}.
\] (2.63b-2.63e)

Finally, the remaining metric potential \( \alpha \) is given by the solution of an ordinary differential equation at each given coordinate \( s \):

\[
\alpha_{s,\mu} = -\frac{1}{2}(\tilde{\rho}_{s,\mu} + \gamma_{s,\mu}) - \left\{ (1 - \mu^2) [1 + s(1-s)\gamma_{s,\mu}]^2 + [\mu - (1 - \mu^2)\gamma_{s,\mu}]^2 \right\}^{-1}
\times \left[ \frac{1}{2} s(1-s) [(1-s)\gamma_{s,\mu}, s] + s^2 (1-s)^2 \gamma_{s,\mu} - [(1 - \mu^2)\gamma_{s,\mu}]_{,,\mu}
\right.
\left. - \gamma_{s,\mu}[-m + (1 - \mu^2)\gamma_{s,\mu}] \right\} [-\mu + (1 - \mu^2)\gamma_{s,\mu}]
\]
\[
+ \frac{1}{4} [s^2(1-s)^2(\tilde{\rho}_{s,\mu} + \gamma_{s,\mu})^2 - (1 - \mu^2)(\tilde{\rho}_{s,\mu} + \gamma_{s,\mu})^2] [-\mu + (1 - \mu^2)\gamma_{s,\mu}]
\]
\[
- s(1-s)(1-\mu^2) \left[ \frac{1}{2} (\tilde{\rho}_{s,\mu} + \gamma_{s,\mu}) (\tilde{\rho}_{s,\mu} + \gamma_{s,\mu}) + \gamma_{s,\mu} + \gamma_{s,\mu} \right] [1 + s(1-s)\gamma_{s,\mu}]
\]
\[
+ s(1-s)\mu \gamma_{s,\mu} [1 + s(1-s)\gamma_{s,\mu}]
\]
\[
+ \frac{1}{4} (1 - \mu^2) e^{-2\tilde{\rho}} \left[ \frac{1}{2} \frac{s^3}{1-s} (1 - \mu^2) \hat{\omega}_{s,\mu} [1 + s(1-s)\gamma_{s,\mu}]
\right.
\]
\[
- \left[ s^4 \hat{\omega}_{s,\mu} - \frac{s^2}{(1-s)^2} (1 - \mu^2) \hat{\omega}_{s,\mu} [-\mu + (1 - \mu^2)\gamma_{s,\mu}] \right] \right] \right],
\] (2.64)

where the following transformations were used:

\[
\tilde{\rho} := \ln \frac{\chi^2}{B} = \nu - \gamma, \tag{2.65a}
\]
\[
\hat{\nu} := \nu/r_{\omega}^2, \tag{2.65b}
\]
\[
\hat{\gamma} := \gamma/r_{\omega}^2, \tag{2.65c}
\]
\[
\hat{\omega} := r_{\omega} \omega. \tag{2.65d}
\]
2.4 Torus properties

The gravitational mass of the torus is given by

\[ M_T = \int (-2T_t^t + T_i^i) \sqrt{-g} \, d^3x \]
\[ = 4\pi r^3 \int_0^1 \frac{s^2}{(1-s)^4} \, ds \int_0^1 d\mu B e^{2\alpha} \left\{ \frac{\epsilon + p}{1 - v^2} \left[ 1 + v^2 + \frac{2sv}{1 - s} \sqrt{1 - \mu^2} \frac{B}{\lambda} \right] + 2p \right\}. \]  
(2.66)

The rest mass of the torus is

\[ M_0 = \int \rho u_t \sqrt{-g} \, d^3x, \]
\[ = 4\pi r^3 \int_0^1 \frac{s^2}{(1-s)^4} \, ds \int_0^1 d\mu B e^{2\alpha} \frac{\rho}{\sqrt{1 - v^2}}. \]  
(2.67)

The internal energy of the torus is

\[ U_T = \int (\epsilon - \rho) u_t \sqrt{-g} \, d^3x, \]
\[ = 4\pi r^3 \int_0^1 \frac{s^2}{(1-s)^4} \, ds \int_0^1 d\mu B e^{2\alpha} \frac{\epsilon - \rho}{\sqrt{1 - v^2}}. \]  
(2.68)

The specific angular momentum of the torus is

\[ J_T = \int T_t^t \phi \sqrt{-g} \, d^3s, \]
\[ = 4\pi r^4 \int_0^1 \frac{s^3}{(1-s)^5} \, ds \int_0^1 d\mu \sqrt{1 - \mu^2} \frac{B^2 e^{2\alpha}}{\lambda^2} \frac{\epsilon + p}{1 - v^2} \frac{v}{1 - v^2}. \]  
(2.69)

The rotational kinetic energy of the torus is

\[ T_T = \frac{1}{2} \int \Omega T_t^t \phi \sqrt{-g} \, d^3s, \]
\[ = 2\pi r^4 \int_0^1 \frac{s^3}{(1-s)^5} \, ds \int_0^1 d\mu \sqrt{1 - \mu^2} \frac{B^2 e^{2\alpha}}{\lambda^2} \frac{\epsilon + p}{1 - v^2} \frac{v \Omega}{1 - v^2}. \]  
(2.70)
Chapter 3

Numerical method and various tori cases.

3.1 Numerical framework

The numerical code starts by constructing an AJS torus around a Schwarzschild black hole in the fixed spacetime of isotropic coordinates.

\[ ds^2 = -\left[1 - \frac{M}{2r}\right] dt^2 + \left(a + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \]

which serves as an initial trial solution for the iterative process. The equilibrium model of the torus is determined by specifying the polytropic constant \(K\), the polytropic index \(N\), the outer radius of the torus \(r_e = r_{out}\) and the two parameters that define the specific angular momentum, the constant \(k\) and the slope \(q\). The whole calculation is done on a grid using compactified coordinates \(s\) and \(\mu = \cos \theta\) in order to be able to apply boundary conditions at infinity. First, we calculate the components of the metric tensor from (3.1) and from those the four metric functions \(\alpha, B(\gamma), \lambda(\nu)\) and \(\omega\), which define the curvature of the spacetime. Then, we calculate the physical quantities that define the fluid of the torus, namely, the distribution of the specific angular momentum from (2.23), the angular velocity from (2.7) and the time component of the 4-velocity from (2.10). The effective potential of the gravitational field is computed from (2.20) where the integral is calculated for every grid point using Simpson’s rule. We use interpolation to calculate \(W_{out}\) and then we obtain the values of enthalpy from (2.31). From the enthalpy we derive the values of the mass density, the pressure and the energy density of the torus. The set of these values define a unique model of an AJS torus. The physical quantities of that model will be used as an input for the self-gravitating torus.
During the iterative process between the field equations (represented by the four metric functions) and the equation of hydrostationary equilibrium, some of the properties are kept fixed. These are the location of the black hole horizon $r_0$ (or $s_0$ in compactified coordinates), the parameters in the specific angular momentum law and in the equation of state and the value of the enthalpy at the point of maximum density. The iteration starts by summoning the previously calculated metric functions and the physical quantities of the fluid. From those quantities we calculate the respective values at specific points, such as the point of maximum density of the torus and the point of the and of the torus (outer radius). These specific point values are going to be used in the calculation of the outer radius $r_{\text{out}}$.

From equation (2.30), for $H = H_{\text{max}}$ we have

$$
H_{\text{out}} - H_{\text{max}} = -\ln(-u_t, \text{out}) + \int_{\text{out}}^{l} \frac{\Omega}{1 - \Omega l} dl + \ln(-u_t, \text{max}) - \int_{\text{out}}^{l_{\text{max}}} \frac{\Omega}{1 - \Omega l} dl.
$$

From equation (2.12), for $u_{t,\text{max}}$ and $u_{t,\text{out}}$ we have

$$
\begin{align*}
    u_t, \text{max} &= -\frac{1}{1 - \Omega_{\text{max}}l_{\text{max}}} \frac{1}{u_t^l}, \\
    u_t, \text{out} &= -\frac{1}{1 - \Omega_{\text{out}}l_{\text{out}}} \frac{1}{u_t^l}.
\end{align*}
$$

Taking the logarithm of the above we have

$$
\ln \left( \frac{-u_t, \text{max}}{-u_t, \text{out}} \right) = \ln \left( \frac{\frac{1}{1 - \Omega_{\text{max}}l_{\text{max}}} \frac{1}{u_t^{l_{\text{max}}}}}{\frac{1}{1 - \Omega_{\text{out}}l_{\text{out}}} \frac{1}{u_t^{l_{\text{out}}}}} \right) = \ln \left( \frac{1 - \Omega_{\text{out}}l_{\text{out}}}{1 - \Omega_{\text{max}}l_{\text{max}}} \frac{u_t^{l_{\text{out}}}}{u_t^{l_{\text{max}}}} \right) = \ln \left( \frac{1 - \Omega_{\text{out}}l_{\text{out}}}{1 - \Omega_{\text{max}}l_{\text{max}}} \frac{1 - v_{\text{max}}^2}{e^{-\nu_{\text{out}}} \sqrt{1 - v_{\text{out}}^2}} \right) = \ln \left( \frac{1 - \Omega_{\text{out}}l_{\text{out}}}{1 - \Omega_{\text{max}}l_{\text{max}}} \frac{1 - v_{\text{max}}^2}{e^{-\nu_{\text{max}}} \sqrt{1 - v_{\text{max}}^2}} \right) = \ln \left( \frac{1 - \Omega_{\text{out}}l_{\text{out}}}{1 - \Omega_{\text{max}}l_{\text{max}}} \right) + \ln \left( \frac{1 - v_{\text{max}}^2}{1 - v_{\text{out}}^2} \right)^{1/2} + (-\nu_{\text{out}} + \nu_{\text{max}}). \right)
Chapter 3. Numerical method and various tori cases.

Since $H_{\text{out}} = 0$

$$-H_{\text{max}} = \ln \left( \frac{1 - \Omega_{\text{out}} l_{\text{out}}}{1 - \Omega_{\text{max}} l_{\text{max}}} \right) + \frac{1}{2} \ln \left( \frac{1 - v_{\text{out}}^2}{1 - v_{\text{out}}^2} \right) + (-\nu_{\text{out}} + \nu_{\text{max}}) - \int_{l_{\text{out}}}^{l_{\text{max}}} \frac{\Omega}{1 - \Omega} dl$$

$$= \ln \left( \frac{1 - \Omega_{\text{out}} l_{\text{out}}}{1 - \Omega_{\text{max}} l_{\text{max}}} \right) + \frac{1}{2} \ln \left( \frac{1 - v_{\text{out}}^2}{1 - v_{\text{out}}^2} \right) + (-r_e \dot{\nu}_{\text{out}} + r_e^2 \nu_{\text{max}}) - \int_{l_{\text{out}}}^{l_{\text{max}}} \frac{\Omega}{1 - \Omega} dl$$

so that

$$-H_{\text{max}} + \int_{l_{\text{out}}}^{l_{\text{max}}} \frac{\Omega}{1 - \Omega} dl = \ln \left( \frac{1 - \Omega_{\text{out}} l_{\text{out}}}{1 - \Omega_{\text{max}} l_{\text{max}}} \right) + \frac{1}{2} \ln \left( \frac{1 - v_{\text{out}}^2}{1 - v_{\text{out}}^2} \right) + r_e^2 (-\dot{\nu}_{\text{out}} + \dot{\nu}_{\text{max}}),$$

$$-H_{\text{max}} + \int_{l_{\text{out}}}^{l_{\text{max}}} \frac{\Omega}{1 - \Omega} dl - \ln \left( \frac{1 - \Omega_{\text{out}} l_{\text{out}}}{1 - \Omega_{\text{max}} l_{\text{max}}} \right) - \frac{1}{2} \ln \left( \frac{1 - v_{\text{out}}^2}{1 - v_{\text{out}}^2} \right) = r_e^2 (-\dot{\nu}_{\text{out}} + \dot{\nu}_{\text{max}}).$$

Then

$$r_e = \left[ \frac{-H_{\text{max}} - \int_{l_{\text{out}}}^{l_{\text{max}}} \frac{\Omega}{1 - \Omega} dl - \ln \left( \frac{1 - \Omega_{\text{out}} l_{\text{out}}}{1 - \Omega_{\text{max}} l_{\text{max}}} \right) - \frac{1}{2} \ln \left( \frac{1 - v_{\text{out}}^2}{1 - v_{\text{out}}^2} \right)}{-\dot{\nu}_{\text{out}} + \dot{\nu}_{\text{max}}} \right]^{1/2}$$

$$= \left[ \frac{-H_{\text{max}} - \int_{l_{\text{out}}}^{l_{\text{max}}} \frac{\Omega}{1 - \Omega} dl - \frac{1}{2} \ln \left( \frac{1 - \Omega_{\text{out}} l_{\text{out}}}{1 - \Omega_{\text{max}} l_{\text{max}}} \right)}{-\dot{\nu}_{\text{out}} + \dot{\nu}_{\text{max}}} \right]^{1/2}$$

and finally

$$r_{\text{out}} \equiv r_e = \sqrt{\frac{1}{\dot{\nu}_{\text{max}} - \dot{\nu}_{\text{out}}} \left[ -H_{\text{max}} - \int_{l_{\text{out}}}^{l_{\text{max}}} \frac{\Omega}{1 - \Omega} dl - \frac{1}{2} \ln \left( \frac{1 - \Omega_{\text{out}} l_{\text{out}}}{1 - \Omega_{\text{max}} l_{\text{max}}} \right) \right]^{1/2}}.$$

We use the newly calculated $r_{\text{out}}$ in deriving the new values for $\Omega, v^2$ and $u_t$. From those, we derive the new effective potential and the new enthalpy and then the new mass density, pressure and energy density using the same equations mentioned before. Finally, we calculate the new values for the four metric functions using equations (2.61), (2.62), (2.63), (2.64). The iterative process resumes from the beginning, until the relative error in $r_{\text{out}}$ becomes smaller than a specific value.
3.2 Detailed approach to the constructed models

We are going to present some of the models created by the code mentioned in the previous section and their respective characteristics. All the models were created using a $501 \times 251$ grid in the $s$ and $\mu$ coordinates.

3.2.1 Unstable models

3.2.1.1 Totally unstable configurations.

The main characteristic of an unstable torus is the fact that the value of the effective potential at the inner radius is a lot smaller than its value at the outer radius. Then, there is no barrier to stop the fluid from flowing into the black hole. The effective potential of such a torus can be seen in Figure 3.1 and the isopotential contours can be seen in Figure 3.2. In the initial AJS model the specific angular momentum is given by $l_{\text{eq}}^\text{AJS} = kR^q$, where $k = 1.7321$ and $q = 0.35$ were given as input trial values. The outer radius of the AJS torus and the constant in the equation of state (EOS) are also given as input ($r_{\text{out}}^\text{AJS}/M_{\text{BH}} = 60$, $K_{\text{EOS}}^\text{AJS}/M^2/N = 0.3$). The constant in the specific angular momentum in the self-gravitating torus is $k/M_{\text{BH}} = 1.699$, the outer radius is $r_{\text{out}}/M_{\text{BH}} = 62$ and has a ratio of $r_{\text{out}}/h_0 = 124$ with respect to the black hole horizon $h_0/M_{\text{BH}} = 0.49$. The mass of the torus is $M_{\text{T}}/M_{\text{BH}} = 6.51 \cdot 10^{-4}$ which is quite expected if we consider the fact that the torus has such a small volume to fill. The inner edge of the torus is at $r_{\text{in}}/M_{\text{BH}} = 12.63$ and the point of maximum density is at $r_{\text{max}}/M_{\text{BH}} = 15.48$. More detailed information about the constructed self-gravitating torus can be found in Table 3.1. Disks such as this one and in general, tori with the value of the effective potential at the inner point of the torus lower than the one at the outer point of the torus, cannot be in equilibrium since there is nothing to stop the gas of the torus to flow into the black hole.

In Figure 3.3 we present the distribution of the four metric functions with respect to the non-dimensional compactified coordinate $s$. For the metric functions $\alpha$, $B$ and $\lambda$ the distribution for the AJS torus is also shown with a dashed line. The potential $\omega$ in the AJS torus is $\omega = 0$, since we have a Schwarzschild black hole. Since the mass of the torus is significantly smaller than the mass of the black hole, the curvature of the spacetime is not affected strongly by the presence of the mass of the torus and the metric functions seem unchanged. However, if we examine the plots closer, there is distinction between the two lines.
Figure 3.1 The effective gravitational potential for an unstable torus.

Table 3.1 Detailed information about the representative model of an unstable torus.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizon radius</td>
<td>$h_0/M_{BH}$</td>
<td>0.490415</td>
</tr>
<tr>
<td>Inner torus radius</td>
<td>$r_{in}/M_{BH}$</td>
<td>12.6321</td>
</tr>
<tr>
<td>Radius of maximum density</td>
<td>$r_{max}/M_{BH}$</td>
<td>15.4833</td>
</tr>
<tr>
<td>Outer torus radius</td>
<td>$r_e/M_{BH}$</td>
<td>60.8176</td>
</tr>
<tr>
<td>Maximum density</td>
<td>$\rho_{max} * M_{BH}^2$</td>
<td>$4.19533 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>Torus mass</td>
<td>$M_T/M_{BH}$</td>
<td>6.51614 $\cdot 10^{-4}$</td>
</tr>
<tr>
<td>Asymptotic spacetime mass</td>
<td>$M/M_{BH}$</td>
<td>0.980621</td>
</tr>
<tr>
<td>Black hole mass</td>
<td>$M_{BH}/M_{BH}$</td>
<td>1.00000</td>
</tr>
<tr>
<td>Black hole Komar charge</td>
<td>$M_{BH}/M_{BH}$</td>
<td>0.979969</td>
</tr>
<tr>
<td>Torus rest mass</td>
<td>$M_0/M_{BH}$</td>
<td>6.37312 $\cdot 10^{-4}$</td>
</tr>
<tr>
<td>Torus internal energy</td>
<td>$U_T/M_{BH}$</td>
<td>8.90475 $\cdot 10^{-7}$</td>
</tr>
<tr>
<td>Torus rotational energy</td>
<td>$T_T/M_{BH}$</td>
<td>1.22328 $\cdot 10^{-5}$</td>
</tr>
<tr>
<td>Torus gravitational potential energy</td>
<td>$W_T/M_{BH}$</td>
<td>$-2.00297 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>Torus rotational to potential energy</td>
<td>$T/</td>
<td>W</td>
</tr>
<tr>
<td>Torus specific angular momentum</td>
<td>$J_T/M_{BH}^2$</td>
<td>3.44025 $\cdot 10^{-3}$</td>
</tr>
</tbody>
</table>
Figure 3.2 The isopotential contours of an unstable torus.
Chapter 3. Numerical method and various tori cases.
Figure 3.3 The four metric functions of an unstable torus in comparison with the ones in the AJS torus.
3.2.1.2 Configurations between the totally unstable limit and the cusp.

In this category we have another torus whose value of the effective potential $W$ at the inner point of the torus is less than the one at the outer point of the torus. In the initial AJS model, the specific angular momentum is given by $l_{eq}^{AJS} = k R^q$, where $k = 2.5$ and $q = 0.2$ were given as input values into the program. The outer radius of the AJS torus is also given as input ($r_{out}^{AJS}/M_{BH} = 60$). The constant in the specific angular momentum in the self-gravitating torus is $k/M_{BH} = 2.56$ and the outer radius is $r_{out}/M_{BH} = 58.8$ with a ratio $r_{out}/h_0 = 124$ with respect to the black hole horizon $h_0/M_{BH} = 0.485$. The mass of the torus is $M_T/M_{BH} = 0.147$. The inner edge of the torus is in $r_{in}/M_{BH} = 4.52$ and the point of maximum density is in $r_{max}/M_{BH} = 13.71$. More detailed information about the constructed self-gravitating torus can be seen in Table 3.2. As in the previous case, such tori also cannot be in stable equilibrium. However, in this case there is a small barrier in the effective potential, which could allow for a torus to stabilize through time evolution, even though it will have lost some of its initial mass.

\[ W \]

**Figure 3.4** The effective potential of a torus between the totally unstable and cusp limits.

In Figure3.6 we present the distribution of the four metric functions with respect to the nondimensional compactified coordinate $s$. For the metric functions $\alpha$, $B$ and $\lambda$ the distribution for the AJS torus is also present with a dashed line. Since the mass of the
Chapter 3. Numerical method and various tori cases.

Figure 3.5 The isopotential contours of a torus between the totally unstable and cusp limits.

torus is $M_T = 0.147/M_{BH}$ the spacetime is affected, meaning that the curvature of the spacetime takes into account the mass of the torus as well as the mass of the black hole.
Table 3.2 Detailed information about the torus between the totally unstable stable and the cusp limits.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizon radius</td>
<td>$h_0/M_{BH}$</td>
<td>0.485178</td>
</tr>
<tr>
<td>Inner torus radius</td>
<td>$r_{in}/M_{BH}$</td>
<td>4.52830</td>
</tr>
<tr>
<td>Radius of maximum density</td>
<td>$r_{max}/M_{BH}$</td>
<td>13.7122</td>
</tr>
<tr>
<td>Outer torus radius</td>
<td>$r_e/M_{BH}$</td>
<td>60.1682</td>
</tr>
<tr>
<td>Maximum density</td>
<td>$\rho_{max} \times M_{BH}^2$</td>
<td>6.38782 · 10^{-6}</td>
</tr>
<tr>
<td>Torus mass</td>
<td>$M_T/M_{BH}$</td>
<td>0.147698e</td>
</tr>
<tr>
<td>Asymptotic spacetime mass</td>
<td>$M/M_{BH}$</td>
<td>1.11849</td>
</tr>
<tr>
<td>Black hole mass</td>
<td>$M_{BH}/M_{BH}$</td>
<td>1.00000</td>
</tr>
<tr>
<td>Black hole Komar charge</td>
<td>$M_{BH}/M_{BH}$</td>
<td>0.970794</td>
</tr>
<tr>
<td>Torus rest mass</td>
<td>$M_0/M_{BH}$</td>
<td>0.143456</td>
</tr>
<tr>
<td>Torus internal energy</td>
<td>$U_T/M_{BH}$</td>
<td>1.08406 · 10^{-3}</td>
</tr>
<tr>
<td>Torus rotational energy</td>
<td>$T_T/M_{BH}$</td>
<td>3.15116 · 10^{-3}</td>
</tr>
<tr>
<td>Torus gravitational potential energy</td>
<td>$W_T/M_{BH}$</td>
<td>-2.91991 · 10^{-2}</td>
</tr>
<tr>
<td>Torus rotational to potential energy</td>
<td>$T/</td>
<td>W</td>
</tr>
<tr>
<td>Torus specific angular momentum</td>
<td>$J_T/M_{BH}^2$</td>
<td>0.682654</td>
</tr>
</tbody>
</table>
Figure 3.6 The four metric functions for a torus between the totally unstable and the cusp limits in comparison with the ones in the AJS torus.
3.2.2 Models at the cusp limit

The next type of model we describe consists of a torus which forms a cusp at its inner radius. Such models have their characteristic cusp because the value of the effective potential $W$ at the local maximum is equal to its value at the outer radius of the torus. As a result, the torus fills completely its “Roche lobe”. The potential of a torus with cusp can be seen in Figure 3.7 and the isopotential contours in Figure 3.8. The model is constructed using a polytropic equation of state with index $N = 3$ and constant $K_{EoS}^{AJS} = 0.2$. The initial AJS model has its specific angular momentum given by $l_{eq}^{AJS} = kr^q$, where $k = 2.5217$ and $q = 0.19318$ were given as input. The outer radius of the AJS torus is also given as input ($r_{out}^{AJS}/M_{BH} = 60$). The constant in the specific angular momentum in the self-gravitating torus is $k/M_{BH} = 2.63$. In the self-gravitating torus the outer radius is $r_{out}/M_{BH} = 59.86$ and has a ratio $r_{out}/h_0 = 124$ with respect to the black hole horizon $h_0/M_{BH} = 0.482$. The mass of the torus is $M_T/M_{BH} = 0.221$ which means that the torus is quite heavy and its self-gravity cannot be ignored. The inner edge of the torus is at $r_{in}/M_{BH} = 4.23$ and the point of maximum density is at $r_{max}/M_{BH} = 14.3$. More detailed information about the constructed self-gravitating torus can be seen in Table 3.3.

![Figure 3.7 The effective potential of a torus filling exactly its Roche lobe.](image)

In Figure 3.9 we present the distribution of the four metric functions with respect to the non-dimensional compactified coordinate $s$. For the metric functions $\alpha$, $B$ and $\lambda$ the
distribution for the AJS torus is also present with a dashed line. In this case, the mass of the torus is $M_T = 0.22/M_{BH}$ and the metric functions of the self-gravitating torus have different distribution compared to the metric functions of the AJS torus.

Figure 3.8 The isopotential contours of a torus filling exactly its Roche lobe.
Table 3.3 Detailed information about the torus in the cusp limit.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizon radius</td>
<td>$h_0/M_{\text{BH}}$</td>
</tr>
<tr>
<td>Inner torus radius</td>
<td>$r_{\text{in}}/M_{\text{BH}}$</td>
</tr>
<tr>
<td>Radius of maximum density</td>
<td>$r_{\text{max}}/M_{\text{BH}}$</td>
</tr>
<tr>
<td>Outer torus radius</td>
<td>$r_e/M_{\text{BH}}$</td>
</tr>
<tr>
<td>Maximum density</td>
<td>$\rho_{\text{max}} \times M_{\text{BH}}^2$</td>
</tr>
<tr>
<td>Torus mass</td>
<td>$M_T/M_{\text{BH}}$</td>
</tr>
<tr>
<td>Asymptotic spacetime mass</td>
<td>$M/M_{\text{BH}}$</td>
</tr>
<tr>
<td>Black hole mass</td>
<td>$M_{\text{BH}}/M_{\text{BH}}$</td>
</tr>
<tr>
<td>Black hole Komar charge</td>
<td>$M_H/M_{\text{BH}}$</td>
</tr>
<tr>
<td>Torus rest mass</td>
<td>$M_0/M_{\text{BH}}$</td>
</tr>
<tr>
<td>Torus internal energy</td>
<td>$U_T/M_{\text{BH}}$</td>
</tr>
<tr>
<td>Torus rotational energy</td>
<td>$T_T/M_{\text{BH}}$</td>
</tr>
<tr>
<td>Torus gravitational potential energy</td>
<td>$W_T/M_{\text{BH}}$</td>
</tr>
<tr>
<td>Torus rotational to potential energy</td>
<td>$T/</td>
</tr>
<tr>
<td>Torus specific angular momentum</td>
<td>$J_T/M_{\text{BH}}^2$</td>
</tr>
</tbody>
</table>
Chapter 3. *Numerical method and various tori cases.*

![Graphs of a and B vs. s](image)
Figure 3.9 The four metric functions for a torus in the cusp limit in comparison with the ones in the AJS torus.
3.2.3 Beyond the cusp limit

In this category, have tori whose value of the effective potential at the local maximum is greater than the one at the outer point and that value is still less than zero. The effective potential of a representative torus can be seen in Figure 3.10 and its isopotential contours in Figure 3.11. In the initial AJS model its specific angular momentum is given by \( l_{eq}^{AJS} = kR^q \), where \( k = 2.57 \) and \( q = 0.2 \) were given as input values. The outer radius of the AJS torus is also given as input \( (r_{out}^{AJS}/M_{BH} = 60) \). The constant in the specific angular momentum in the self-gravitating torus is \( k/M_{BH} = 2.58 \) and the outer radius is \( r_{out}/M_{BH} = 60.27 \) having a ratio of \( r_{out}/h_0 = 124 \) with the black hole horizon \( h_0/M_{BH} = 0.488 \). The mass of the torus is \( M_T/M_{BH} = 0.07 \) which means that its contribution to the curvature of the spacetime is appreciable. The inner edge of the torus is at \( r_{in}/M_{BH} = 5.88 \) and the point of maximum density is at \( r_{max}/M_{BH} = 14.8 \). More detailed information about the constructed self-gravitating torus can be seen in Table 3.4.

![Figure 3.10](image)

**Figure 3.10** The effective potential of a torus beyond the cusp limit.

In Figure 3.12 we present the distribution of the four metric functions with respect to the nondimensional compactified coordinate \( s \). For the metric functions \( \alpha, B \) and \( \lambda \) the distribution for the AJS torus is also present with a dashed line. Even though the mass of the torus is not as high as in the previous two cases, we can still see how the metric functions are affected when we take into consideration the extra mass of the torus.
Chapter 3. *Numerical method and various tori cases.*

**Figure 3.11** The isopotential contours of a torus beyond the cusp limit.

**Table 3.4** Detailed information about the torus beyond the cusp limit.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol/Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizon radius</td>
<td>$h_0/M_{BH}$</td>
<td>0.488055</td>
</tr>
<tr>
<td>Inner torus radius</td>
<td>$r_{in}/M_{BH}$</td>
<td>5.87921</td>
</tr>
<tr>
<td>Radius of maximum density</td>
<td>$r_{max}/M_{BH}$</td>
<td>14.8095</td>
</tr>
<tr>
<td>Outer torus radius</td>
<td>$r_e/M_{BH}$</td>
<td>60.5249</td>
</tr>
<tr>
<td>Maximum density</td>
<td>$\rho_{max} \cdot M_{BH}^2$</td>
<td>2.92289 $\cdot 10^{-6}$</td>
</tr>
<tr>
<td>Torus mass</td>
<td>$M_T/M_{BH}$</td>
<td>0.0699297</td>
</tr>
<tr>
<td>Asymptotic spacetime mass</td>
<td>$M/M_{BH}$</td>
<td>1.04583</td>
</tr>
<tr>
<td>Black hole mass</td>
<td>$M_{BH}/M_{BH}$</td>
<td>1.00000</td>
</tr>
<tr>
<td>Black hole Komar charge</td>
<td>$M_H/M_{BH}$</td>
<td>0.975897</td>
</tr>
<tr>
<td>Torus rest mass</td>
<td>$M_0/M_{BH}$</td>
<td>0.0680403</td>
</tr>
<tr>
<td>Torus internal energy</td>
<td>$U_T/M_{BH}$</td>
<td>3.93070 $\cdot 10^{-4}$</td>
</tr>
<tr>
<td>Torus rotational energy</td>
<td>$T_T/M_{BH}$</td>
<td>1.40919 $\cdot 10^{-3}$</td>
</tr>
<tr>
<td>Torus gravitational potential energy</td>
<td>$W_T/M_{BH}$</td>
<td>$-2.40155 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>Torus rotational to potential energy</td>
<td>$T/</td>
<td>W</td>
</tr>
<tr>
<td>Torus specific angular momentum</td>
<td>$J_T/M_{RH}^2$</td>
<td>0.329766</td>
</tr>
</tbody>
</table>
Chapter 3. Numerical method and various tori cases.
Figure 3.12 The four metric functions for a torus beyond the cusp limit in comparison with the ones in the AJS torus.
3.2.4 Models at the $W_{\text{peak}} = 0$ limit

Disks that belong to this category have a local maximum in the effective potential to zero. The effective potential of a representative model can be seen in Figure 3.13 and the respective isopotential contours in Figure 3.14. In the initial AJS model, the specific angular momentum is given by $l_{\text{eq}}^{\text{AJS}} = kr^q$, where $k = 2.6043$ and $q = 0.178$ were given as input values. The outer radius of the AJS torus is also given as input ($r_{\text{out}}^{\text{AJS}}/M_{\text{BH}} = 60$). The constant in the specific angular momentum in the self-gravitating torus is $k/M_{\text{BH}} = 2.807$ and the outer radius is $r_{\text{out}}/M_{\text{BH}} = 55.15$ with a ratio of $r_{\text{out}}/h_0 = 124$ with respect to the black hole horizon $h_0/M_{\text{BH}} = 0.479$. The mass of the torus is $M_T/M_{\text{BH}} = 0.332$, which means that its contribution to the curvature of the spacetime is important. The inner edge of the torus is at $r_{\text{in}}/M_{\text{BH}} = 6.014$ and the point of maximum density is at $r_{\text{max}}/M_{\text{BH}} = 15.84$. More detailed information about the constructed self-gravitating torus can be seen in Table 3.5.

![Figure 3.13](image)

**Figure 3.13** The effective potential of a torus in the $W_{\text{peak}} = 0$ limit.

In Figure 3.15 we present the distribution of the four metric functions with respect to the non-dimensional compactified coordinate $s$. For the metric functions $\alpha$, $B$ and $\lambda$ the distribution for the AJS torus is also present with a dashed line. In this case, the mass of the torus is quite high compared with the mass of the black hole and the effect of that extra mass on the spacetime can be seen by the changes in the distribution of the metric functions.
**Figure 3.14** The isopotential contours of a torus in the $W_{\text{peak}} = 0$ limit.

**Table 3.5** Detailed information about the torus at the $W_{\text{peak}} = 0$ limit.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizon radius</td>
<td>$h_0/M_{\text{BH}}$</td>
<td>0.479401</td>
</tr>
<tr>
<td>Inner torus radius</td>
<td>$r_{\text{in}}/M_{\text{BH}}$</td>
<td>6.01472</td>
</tr>
<tr>
<td>Radius of maximum density</td>
<td>$r_{\text{max}}/M_{\text{BH}}$</td>
<td>15.8418</td>
</tr>
<tr>
<td>Outer torus radius</td>
<td>$r_e/M_{\text{BH}}$</td>
<td>59.4517</td>
</tr>
<tr>
<td>Maximum density</td>
<td>$\rho_{\text{max}} \cdot M_{\text{BH}}^2$</td>
<td>1.13064 \cdot 10^{-5}</td>
</tr>
<tr>
<td>Torus mass</td>
<td>$M_T/M_{\text{BH}}$</td>
<td>0.332224</td>
</tr>
<tr>
<td>Asymptotic spacetime mass</td>
<td>$M/M_{\text{BH}}$</td>
<td>1.29229</td>
</tr>
<tr>
<td>Black hole mass</td>
<td>$M_{\text{BH}}/M_{\text{BH}}$</td>
<td>1.00000</td>
</tr>
<tr>
<td>Black hole Komar charge</td>
<td>$M_H/M_{\text{BH}}$</td>
<td>0.960064</td>
</tr>
<tr>
<td>Torus rest mass</td>
<td>$M_0/M_{\text{BH}}$</td>
<td>0.323436</td>
</tr>
<tr>
<td>Torus internal energy</td>
<td>$U_T/M_{\text{BH}}$</td>
<td>3.17076 \cdot 10^{-3}</td>
</tr>
<tr>
<td>Torus rotational energy</td>
<td>$T_T/M_{\text{BH}}$</td>
<td>6.86422 \cdot 10^{-3}</td>
</tr>
<tr>
<td>Torus gravitational potential energy</td>
<td>$W_T/M_{\text{BH}}$</td>
<td>-4.11833 \cdot 10^{-2}</td>
</tr>
<tr>
<td>Torus rotational to potential energy</td>
<td>$T/</td>
<td>W</td>
</tr>
<tr>
<td>Torus specific angular momentum</td>
<td>$J_T/M_{\text{BH}}^2$</td>
<td>1.58354</td>
</tr>
</tbody>
</table>
Figure 3.15 The four metric functions for a torus at the $W_{\text{peak}} = 0$ limit.
3.2.5 Approaching the upper limit

In this category we have tori that are not bound to the black hole. The effective gravitational potential has positive values near the black hole, which does not allow the fluid to reach the black hole. In this case, the torus has a shape close to a toroid without the ability to form a cusp. The effective potential of such a torus can be seen in Figure 3.16 and the respective isopotential contours in Figure 3.17. In the initial AJS model the specific angular momentum given by \( l_{eq}^{\text{AJS}} = kR^q \), where \( k = 3 \) and \( q = 0.2 \) were given as input. The outer radius of the AJS torus is given as input \( (r_{out}^{\text{AJS}}/M_{\text{BH}} = 60) \). The constant in the specific angular momentum in the self-gravitating torus is \( k/M_{\text{BH}} = 2.94 \) and the outer radius is \( r_{out}/M_{\text{BH}} = 61.9 \) having a ratio of \( r_{out}/h_0 = 124 \) with respect to the black hole horizon \( h_0/M_{\text{BH}} = 0.49 \). The mass of the torus is \( M_{T}/M_{\text{BH}} = 9.73 \cdot 10^{-4} \) which means that its contribution to the curvature of the spacetime is negligible. The inner edge of the torus is at \( r_{in}/M_{\text{BH}} = 18.7 \) and the point of maximum density is at \( r_{max}/M_{\text{BH}} = 29.8 \). More detailed information about the constructed self-gravitating torus can be seen in Table 3.6. An important property of such tori is that as the specific angular momentum constant increases, the location of maximum density approaches the outer edge of the torus.

![Figure 3.16 The effective potential of a torus approaching the upper limit.](image)
Chapter 3. Numerical method and various tori cases.

Figure 3.17 The isopotential contours of a torus approaching the upper limit.

In Figure 3.18 we present the distribution of the four metric functions with respect to the nondimensional compactified coordinate $s$. For the metric functions $\alpha$, $B$ and $\lambda$ the distribution for the AJS torus is also present with a dashed line.
Table 3.6 Detailed information about the torus approaching the upper limit.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizon radius</td>
<td>$h_0/M_{BH}$</td>
<td></td>
<td>0.490413</td>
</tr>
<tr>
<td>Inner torus radius</td>
<td>$r_{in}/M_{BH}$</td>
<td></td>
<td>18.7273</td>
</tr>
<tr>
<td>Radius of maximum density</td>
<td>$r_{max}/M_{BH}$</td>
<td></td>
<td>29.8006</td>
</tr>
<tr>
<td>Outer torus radius</td>
<td>$r_e/M_{BH}$</td>
<td></td>
<td>60.8174</td>
</tr>
<tr>
<td>Maximum density</td>
<td>$\rho_{max} * M_{BH}^2$</td>
<td></td>
<td>$2.63144 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>Torus mass</td>
<td>$M_T/M_{BH}$</td>
<td></td>
<td>$9.73354 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>Asymptotic spacetime mass</td>
<td>$M/M_{BH}$</td>
<td></td>
<td>0.980931</td>
</tr>
<tr>
<td>Black hole mass</td>
<td>$M_{BH}/M_BH$</td>
<td></td>
<td>1.00000</td>
</tr>
<tr>
<td>Black hole Komar charge</td>
<td>$M_H/M_{BH}$</td>
<td></td>
<td>0.979958</td>
</tr>
<tr>
<td>Torus rest mass</td>
<td>$M_0/M_{BH}$</td>
<td></td>
<td>$9.57871 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>Torus internal energy</td>
<td>$U_T/M_{BH}$</td>
<td></td>
<td>$1.25843 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>Torus rotational energy</td>
<td>$T_T/M_{BH}$</td>
<td></td>
<td>$1.32995 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>Torus gravitational potential energy</td>
<td>$W_T/M_{BH}$</td>
<td></td>
<td>$-2.00412 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>Torus rotational to potential energy</td>
<td>$T/</td>
<td>W</td>
<td>$</td>
</tr>
<tr>
<td>Torus specific angular momentum</td>
<td>$J_T/M_{BH}^2$</td>
<td></td>
<td>$5.72859 \cdot 10^{-3}$</td>
</tr>
</tbody>
</table>
Figure 3.18 The four metric functions for a torus approaching the upper in comparison with the ones in the AJS torus.
3.2.6 The upper limit

In this category there are tori with an high value of specific angular momentum, and extremely low $M_T/M_{BH}$ ratio. Disks like that, mark the limit for the values of the specific angular momentum constant for which a torus can be created with the requested outer radius. As we can see from the isopotential contours in Figure 3.20, the location of maximum density of the torus is very close to its outer edge, which means that a further increase of the specific angular momentum will result in a torus whose location of maximum density will be out of the torus. In the initial AJS model the specific angular momentum given by $l_{eq}^{\text{AJS}} = kr^q$, where $k = 3.475$ and $q = 0.2$ was given as input. The outer radius of the AJS torus is also given as input ($r_{out}^{\text{AJS}}/M_{BH} = 60$). The constant in the specific angular momentum in the self-gravitating torus is $k/M_{BH} = 3.43$ and the outer radius is $r_{out}/M_{BH} = 61.5$, having a ratio of $r_{out}/h_0 = 124$ with respect to the black hole horizon $h_0/M_{BH} = 0.49$. The mass of the torus is $M_T/M_{BH} = 2.17 \cdot 10^{-10}$. The inner edge of the torus is at $r_{in}/M_{BH} = 51.7$ and the point of maximum density is at $r_{max}/M_{BH} = 55.9$. More detailed information about the constructed self-gravitating torus can be seen in Table 3.7.

![Figure 3.19](image) The effective potential of the torus just before the location of maximum density reaches the outer edge of the torus.

In Figure (3.21) we present the distribution of the four metric functions with respect to the nondimensional compactified coordinate $s$. For the metric functions $\alpha, B$ and $\lambda$ the
distribution for the AJS torus is also presented with a dashed line. In this case the mass of the torus is $M_T \approx 10^{-10} M_{BH}$ and the spacetime is not affect at all by its presence.
Table 3.7 Detailed information about the torus just before the location of maximum density reaches the outer edge of the torus.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizon radius</td>
<td>$h_0/M_{BH}$</td>
<td>0.490435</td>
</tr>
<tr>
<td>Inner torus radius</td>
<td>$r_{in}/M_{BH}$</td>
<td>51.7251</td>
</tr>
<tr>
<td>Radius of maximum density</td>
<td>$r_{\text{max}}/M_{BH}$</td>
<td>55.9404</td>
</tr>
<tr>
<td>Outer torus radius</td>
<td>$r_e/M_{BH}$</td>
<td>60.8201</td>
</tr>
<tr>
<td>Maximum density</td>
<td>$\rho_{\text{max}} * M_{BH}^2$</td>
<td>$3.54835 \cdot 10^{-13}$</td>
</tr>
<tr>
<td>Torus mass</td>
<td>$M_T/M_{BH}$</td>
<td>$2.17856 \cdot 10^{-10}$</td>
</tr>
<tr>
<td>Asymptotic spacetime mass</td>
<td>$M/M_{BH}$</td>
<td>0.979994</td>
</tr>
<tr>
<td>Black hole mass</td>
<td>$M_{BH}/M_{BH}$</td>
<td>1.00000</td>
</tr>
<tr>
<td>Black hole Komar charge</td>
<td>$M_H/M_{BH}$</td>
<td>0.979994</td>
</tr>
<tr>
<td>Torus rest mass</td>
<td>$M_0/M_{BH}$</td>
<td>$2.15876 \cdot 10^{-10}$</td>
</tr>
<tr>
<td>Torus internal energy</td>
<td>$U_T/M_{BH}$</td>
<td>$3.92416 \cdot 10^{-15}$</td>
</tr>
<tr>
<td>Torus rotational energy</td>
<td>$T_T/M_{BH}$</td>
<td>$1.90077 \cdot 10^{-12}$</td>
</tr>
<tr>
<td>Torus gravitational potential energy</td>
<td>$W_T/M_{BH}$</td>
<td>$-2.00058 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>Torus rotational to potential energy</td>
<td>$T/</td>
<td>W</td>
</tr>
<tr>
<td>Torus specific angular momentum</td>
<td>$J_T/M_{BH}^2$</td>
<td>$1.65681 \cdot 10^{-9}$</td>
</tr>
</tbody>
</table>
Chapter 3. Numerical method and various tori cases.
Figure 3.21 The four metric functions for a torus just before the location of maximum density reaches the outer edge of the torus.
Chapter 4

The parameter space of equilibrium models

4.1 The subspace for the specific angular momentum parameters $K_{EOS}^{AJS} = 0.2$ and $r_{out}^{AJS} = 39$

We will try to find the subspace of equilibrium models for specific angular momentum parameters and locate the areas where the previous types of tori exist. In Figures 4.1a and 4.1b we display the parameter subspace for models with $K_{EOS}^{AJS} = 0.2$, $N = 3$ and $r_{out}^{AJS} = 39$. On the horizontal axis we have the exponent of the specific angular momentum law and in the vertical axis the respective constant. For each value of the exponent $q$ they exist different torus models, depending on the value of the constant $k$. In Figure 4.1a we plot the parameter subspace for the initial test values used as an input in the numerical code, while in Figure 4.1b we show the corresponding rescaled converged values of the final self-gravitating models.

The whole area is divided into different regions. Between lines $\alpha$ and $\gamma$ (i.e. in area $\beta$ there are unstable tori, similar to the ones described in section 3.2.1. Line $\gamma$ defines tori that form a cusp at their inner edge (section 3.2.2). In area $\delta$ there are tori with local $W_{\text{max}}$ larger than the value of $W$ at the outer edge of the torus (section ). Line $\epsilon$ defines tori with $W_{\text{max}} > 0$ (section 3.2.4). In area $\zeta$ the location of maximum density starts moving towards the outer edge of the torus as the specific angular momentum constant increases (section 3.2.5), until line $\eta$ is reached, where the point of maximum density practically coincides with the outer edge (section 3.2.6). The shape of the effective potential in each of these cases is shown in Figure 4.2.
The tori in this subspace show a great variation of masses ranging from tori as massive as the black hole itself, to tori with a mass of ~ $10^{-10} M_{\text{BH}}$. As a general rule we can say that for a stronger specific angular momentum, we obtain a lighter torus for fixed $K_{\text{EOS}}^{\text{AJS}}$ and $r_{\text{out}}^{\text{AJS}}$. A characteristic of these figures is that the lines defining the tori with cusp and the tori with $W_{\text{max}} = 0$ do not exist for every value of the exponent $q$. The first torus with cusp appears at $q = 0.105$ and the first with $W_{\text{max}} = 0$ appears at $q = 0.08$. We note that in the lower part of Figure 4.1 the line marking the lower limit does not mark at low values of $q$ (see discussion below) the same models for all values of $q$, as in the upper panel, because of an overlap.

In Figures 4.3a and 4.3b we present the values of the mass of the torus for the combinations of the two parameters in the specific angular momentum law. As we can see, for the rescaled values of the specific angular momentum parameters, there are some combinations of the two parameters which covers two different equilibrium models (differing in torus mass). The models in this “overlap” region also have different outer radii and EOS constants.

In Figure 4.4 is a contour plot of the three-dimensional data showing in Figure 4.3. The horizontal and vertical axis are the exponent and the constant of the specific angular momentum law respectively. If we compare it with Figure 4.1b we see that the area covered by the contour plot is larger. For the most part, the rescaled $k_{\text{AJS}}$ decreases along a constant $q$ line as the mass of the torus increases. But for very large torus mass this trend is reversed.

In Figure 4.5a we present the parametric plots of the constant in the specific angular momentum law with respect to the mass of the torus for different values of the exponent $q$. The value of $K_{\text{AJS}}/M_{\text{BH}}$ decreases as the mass of the torus increases, until the torus mass reaches a critical value from which on $K_{\text{AJS}}/M_{\text{BH}}$ increases. Figure 4.5b shows the values of the critical mass with respect to the exponent $q$ for the above set of models. The critical mass is for overlap is in the range $0.14 M_{\text{BH}} < M_{\text{cr}} < 0.18 M_{\text{BH}}$. 
Figure 4.1 The subspace of the specific angular momentum parameters for $K_{\text{EOS}}^{\text{AJS}} = 0.2$ and $r_{\text{out}}^{\text{AJS}} = 39$ models. (a) for the initial trial values of the constant $k$, (b) for the rescaled values.
Figure 4.2 The effective potential for the cases presented in Figures 4.1 for tori with $K_{E0S}^{AJS} = 0.2$ and $r_{out}^{AJS} = 39$. 
Chapter 4. The parameter space of equilibrium models

Figure 4.3 The dependence of the torus mass from the (input and rescaled) specific angular momentum parameters for $K_{EOS}^{AJS} = 0.2$ and $r_{out}^{AJS} = 39$. (a) is for the initial trial values of the constant $k$, while (b) is for the rescaled values.
Figure 4.4 The contour plot of the three-dimensional data. The torus mass as a function of the two specific angular momentum parameters for $K_{EoS}^{AJS} = 0.2$ and $r_{out}^{AJS} = 39$ models.
Figure 4.5 The parametric plot showing the dependence of the specific angular momentum constant from the torus mass and the critical torus mass for different values of slope $q$, for $K_{\text{EOS}}^{\text{AJS}} = 0.2$ and $r_{\text{out}}^{\text{AJS}} = 39$.
Chapter 4. *The parameter space of equilibrium models*

4.2 The subspace for the specific angular momentum parameters \( K_{\text{EOS}}^{\text{AJS}} = 0.2 \) and \( r_{\text{out}}^{\text{AJS}} = 60 \)

Now we will examine the subspace of equilibrium models with the same equation of state constant as before \((K_{\text{EOS}}^{\text{AJS}} = 0.2)\), but with a different input outer radius for the AJS torus \((r_{\text{out}}^{\text{AJS}} = 60)\). The subspace for the input trial values of the specific angular momentum is shown in Figure 4.6a, and the rescaled converged ones for the self-gravitating torus can be seen in Figure 4.6b. Since the outer radius of the torus is larger than the one in the previous case, we obtain heavier tori for the same values of the specific angular momentum parameters.

The initial conditions for this particular subspace of the parametric space results in the creation of some very massive tori with masses close to \( M_T = 200\% M_{\text{BH}} \). As the specific angular momentum increases, the mass of the constructed tori decreases, sometimes reaching values as low as \( \sim 10^{-10} M_{\text{BH}} \). The increase in the torus mass results in the large differences between Figures 4.6a and 4.6b, especially for small values of the specific angular momentum exponent. The line which marks the tori in the cusp limit as well as the line which marks the tori with \( W_{\text{peak}} = 0 \), do not appear until the exponent of the specific angular momentum law becomes equal to \( q = 0.193 \) and \( q = 0.178 \) respectively.

The different lines and areas are marked in the same way as the ones in Figures 4.1a,b and have the same explanations. The distribution of the effective potential for these representative tori can be seen in Figure 4.7.

In Figures 4.8a and 4.8b we present the dependence of the torus mass on the specific angular momentum parameters in three-dimensional plots for models with \( K_{\text{EOS}}^{\text{AJS}} = 0.2 \) and \( r_{\text{out}}^{\text{AJS}} = 60 \). In the horizontal plane we show the aforementioned parameters and on the vertical axis the logarithm of the mass of the torus as a fraction of the black hole mass. Similarly with the previous case studied, we note that there are combinations of the specific angular momentum parameters that result in two tori with different masses. The reason behind that is explained in the previous case. The only difference in this case is that the mass of the tori in general are larger and the area for which the combination of values of the specific angular momentum parameters result in two models with different torus mass, is larger.

Again we will try to find the critical torus mass that causes this overlap. Figure 4.9 shows the contour plot of the three dimensional data. In comparison with Figure 4.6b, we note that the area covered by the contour plot is larger than the area covered by the parametric plots of the constant in the specific angular momentum law with respect to the mass of the torus for each values of the exponent. It appears that a critical value of torus mass must be reached in order to have the aforementioned double mapping.
In Figure 4.10a we present the parametric plots. In Figure 4.10b we see the values of the torus mass which appear to form a threshold for the specific angular momentum constant to start increasing with the torus mass. The critical torus mass is in the range $0.14 M_{\text{BH}} < M_c < 0.21 M_{\text{BH}}$. 
Figure 4.6 The subspace of models for the specific angular momentum parameters for $K_{EoS}^{AJS} = 0.2$ and $r_{out}^{AJS} = 60$. (a) is for the initial trial values of the constant $k$, while (b) is for the rescaled values.
Figure 4.7 The effective potential for the cases presented in Figures 4.6 for tori with $K_{\text{EOS}}^{\text{AJS}} = 0.2$ and $r_{\text{out}}^{\text{AJS}} = 60$. 
Figure 4.8 The dependence of the torus mass on the (input and rescaled) specific angular momentum parameters for $k_{E_0}^{AJS} = 0.2$ and $r_{out}^{AJS} = 60$. (a) is for the initial trial values of the constant $k$, while (b) is for the rescaled values.
Figure 4.9 The torus mass as a function of the two specific angular momentum parameters for $K_{\text{EOS}}^{AJS} = 0.2$ and $r_{\text{out}}^{AJS} = 60$ models.
Chapter 4. *The parameter space of equilibrium models*

Figure 4.10 Parametric plot showing the dependence of the specific angular momentum constant from the torus mass and the critical torus mass for different values of slope $q$, for $K_{\text{EOS}} = 0.2$ and $r_{\text{out}} = 60$. 
4.3 The subspace of the specific angular momentum parameters for $K_{\text{EOS}}^{AJS} = 0.3$ and $r_{\text{out}}^{AJS} = 39$

In this section we will examine the subspace for equilibrium models with $K_{\text{EOS}}^{AJS} = 0.3$ and $r_{\text{out}}^{AJS} = 39$. The increase in the value of the equation of state constant results in a lighter and less dense torus, as it is easily understood from (2.32). As a result, the mass of the torus is changed with almost the same factor as the density of the torus. The subspace of the initial trial values of the specific angular momentum parameters is shown in Figure 4.11a and the subspace of the rescaled converged values is shown in Figure 4.11b. Since the values of the equation of state constant and the outer radius result in lighter tori in general, the heaviest torus from these simulations has a mass $\sim 0.22M_{\text{BH}}$. The torus masses in this case are not as large as in the previous cases, and as a result the corresponding subspaces do not show differences. Both the lines that mark the cusp and the $W_{\text{peak}} = 0$ limits have non-zero values even for constant specific angular momentum.

In Figures 4.13a and 4.13b we can see how the mass of the torus changes with the values of the specific angular momentum parameters for $K_{\text{EOS}}^{AJS} = 0.3$ and $r_{\text{out}}^{AJS}=39$. In contrast with the previously studied cases, there are no combination of the two specific angular momentum parameters for which we have two tori with different masses. As a result of that, the two three-dimensional plots are very similar even though there are still differences in the values of the parameters. The parametric plot of the specific angular momentum constant as a function of the torus mass can be seen in Figure 4.15a and we can see that there are no values of the exponent for which we have critical mass. Nonetheless, we plot the minimum mass values in Figure 4.15b.
Figure 4.11 The subspace of the specific angular momentum parameters for $K_{\text{EOS}}^{\text{AJS}} = 0.3$ and $r_{\text{out}}^{\text{AJS}} = 39$ models. (a) is for the initial trial values of the constant $k$, while (b) is for the rescaled values.
Figure 4.12 The effective potential for the cases presented in Figures 4.11 for tori with $K_{\text{EOS}}^{\text{AJS}} = 0.3$ and $r_{\text{out}}^{\text{AJS}} = 39$. 

Chapter 4. The parameter space of equilibrium models
Figure 4.13 The dependence of the torus mass from the (input and rescaled) specific angular momentum parameters for $K_{\text{EOS}} = 0.3$ and $r_{\text{out}}^{\text{AJS}} = 39$. (a) is for the initial trial values of the constant $k$, while (b) is for the rescaled values.
Figure 4.14 The contour plot of the three-dimensional data for $K_{\text{EOS}}^{\text{KJS}} = 0.3$ and $r_{\text{out}}^{\text{KJS}} = 39$. The torus mass is shown as a function of the two specific angular momentum parameters.
Figure 4.15 Parametric plot showing the dependence of the specific angular momentum constant from the torus mass and the critical torus mass for different values of exponent $q$, for $K_{EoS}^{AJS} = 0.3$ and $r_{out}^{AJS} = 39$. 

(a) 

(b)
4.4 The subspace of the specific angular momentum parameters for $K_{EOS}^{AJS} = 0.3$ and $r_{out}^{AJS} = 60$

This case is similar to the first one we studied, where $K_{EOS}^{AJS} = 0.2$ and $r_{out}^{AJS} = 39$, since only one of the parameters has a value that results in a torus with increased mass. So, in our case $K_{EOS}^{AJS} = 0.3$ and $r_{out}^{AJS} = 60$. In Figures 4.16a and 4.16b we show the subspace for the initial trial values of the specific angular momentum parameters and for the rescaled ones, respectively. In this case, the heaviest torus created has a mass around 105% the mass of the black hole. Disks with cusp do not appear until the specific angular momentum exponent becomes equal to $q = 0.0955$ and the $W_{peak} = 0$ limit appears for the first time for $q = 0.0785$.

As in previous cases, we have pairs of the specific angular momentum parameters corresponding to two tori with different masses, as can be seen in the three-dimensional plots in Figures 4.18a and 4.18b. The area that is covered by the overlap in mass is a little larger than the one in the first case examined ($K_{EOS}^{AJS} = 0.2, r_{out}^{AJS} = 39$). That’s because the changes in the equation of state constant and in the outer radius did not balance each other out, resulting in heavier tori.

The overlap appears when the constructed torus is heavier than a critical value. As we can see in Figure 4.20a the overlap in mass exists for some of the lower values of the exponent $q$. The minimum in those parametric plots gives us the value of the critical mass. In Figure 4.20b we show the minimum mass values for each value of the specific angular momentum exponent. The critical mass is in the range $0.21M_{BH} < M_{cr} < 0.27M_{BH}$. 
Figure 4.16 The subspace of the specific angular momentum parameters for $K_{\text{EOS}}^{\Lambda J S} = 0.3$ and $r_{\text{out}}^{\Lambda J S} = 60$. (a) is for the initial trial values of the constant $k$, while (b) is for the rescaled values.
Figure 4.17 The effective potential for the cases presented in Figures 4.16 for tori with $K_{\text{EOS}}^{\text{AJS}} = 0.3$ and $r_{\text{out}}^{\text{AJS}} = 60$. 
Figure 4.18 The dependence of the torus mass on the (input and rescaled) specific angular momentum parameters for $K_{\text{EOS}}^{\text{AJS}} = 0.3$ and $r_{\text{out}}^{\text{AJS}} = 60$. (a) is for the initial trial values of the constant $k$, while (4.18b) is for the rescaled values.
Figure 4.19 Contour plot of the three-dimensional data showing in figure 4.3. The color scale corresponds to the torus mass.
Figure 4.20 Parametric plot showing the dependence of the specific angular momentum constant from the torus mass and the critical torus mass $M_{cr}$ for overlap, for different values of exponent $q$, for $K_{AJS}^{EOS} = 0.3$ and $r_{AJS}^{out} = 60$. 
Chapter 5

Conclusions and comments

We are going to compare the results presented in the previous Sections. The mass of the torus is determined by the values of the following four parameters: The two parameters that define the angular momentum per unit energy $k$ and $q$, the outer radius of the torus $r_{\text{out}}$ and the constant in the equation of state $K_{\text{EOS}}$. As we have already pointed out, there seems to be a critical mass for the torus which causes an overlapping of models for the same values of the specific angular momentum parameters. These values are presented in Table 5.1.

Constructing models with mass lower than the critical value, an increase in the parameters of the specific angular momentum causes a decrease in the mass of the constructed torus, while for models heavier than the ones in the critical limit an increase in the distribution on the specific angular momentum also results in the construction of heavier tori. For massive tori, the self-gravity of the torus stabilizes the torus for higher distributions of the specific angular momentum. Figures 5.1 and 5.2 show how the outer radius and the constant in the equation of state change with the mass of the constructed torus for the models examined in the previous Sections.

<table>
<thead>
<tr>
<th>$K_{\text{EOS}}$</th>
<th>$r_{\text{out}}^{\text{AJS}}$</th>
<th>$(M/T/M_{\text{BH}})_{\text{min}}$</th>
<th>$(M/T/M_{\text{BH}})_{\text{mean}}$</th>
<th>$(M/T/M_{\text{BH}})_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>39</td>
<td>0.1463</td>
<td>0.1641</td>
<td>0.1814</td>
</tr>
<tr>
<td>0.2</td>
<td>60</td>
<td>0.1435</td>
<td>0.1709</td>
<td>0.2004</td>
</tr>
<tr>
<td>0.3</td>
<td>39</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.3</td>
<td>60</td>
<td>0.2165</td>
<td>0.2452</td>
<td>0.2705</td>
</tr>
</tbody>
</table>

Table 5.1 The minimum, mean and maximum value of the torus mass which causes the overlap in mass.
Chapter 5. Conclusions and comments

(a) $K_{\text{EOS}}^{\text{AJS}} = 0.2$, $r_{\text{out}}^{\text{AJS}} = 39$

(b) $K_{\text{EOS}}^{\text{AJS}} = 0.2$, $r_{\text{out}}^{\text{AJS}} = 50$
Figure 5.1 The changes in the outer radius for the higher values of the torus mass.
Chapter 5. Conclusions and comments

\[(a) \quad K_{\text{EOS}}^{\text{AJS}} = 0.2, \quad r_{\text{out}}^{\text{AJS}} = 39\]

\[(b) \quad K_{\text{EOS}}^{\text{AJS}} = 0.2, \quad r_{\text{out}}^{\text{AJS}} = 50\]
Figure 5.2 The changes in the equation of state constant for the higher values of the torus mass.
The results show that the overlap in mass of the torus is around $0.15M_{BH}$, which suggests that the study of such massive tori should be done taking into consideration the effect of the torus on the curvature of the spacetime. Furthermore, simulations suggest that such tori can be products by the merging between two neutron stars, with a torus to black hole mass ratio up to $\sim 15\%$. It is therefore very important to take self-gravity into account when studying heavy tori as central engines for short gamma-ray-bursts.
Bibliography


