Symplectic Integration of Equations of Motion and Variational Equations for Extrasolar Systems of N - Planets

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Abstract

The aim of this present thesis was the symplectic integration of the equations of motion and the variational equations for a N - body system. In order to implement the integration we use a symplectic 6th order integrator presented by Yoshida. For the integration of variational equations we apply the Tangent Map (TM) method. The algorithm is applied on three Extrasolar systems : GJ 876, HR 8799 and TRAPPIST-1. The Fast Lyapunov Indicator (FLI) of the system is calculated for all systems and it is used as a tool to detect chaotic orbits. We also apply the Angular Momentum Deficit criterion to decide whether this systems are AMD - stable or not.

Περίληψη

Σκοπός αυτής της διατριβής είναι η συμπλεκτική ολοκλήρωση των εξισώσεων της κίνησης και των εξισώσεων των μεταβολών σε ένα σύστημα N - σωμάτων. Για να επιτύχουμε την αριθμητική ολοκλήρωση υλοποιούμε τον αλγόριθμο του συμπλεκτικού ολοκλήρωσης της Yoshida. Για την ολοκλήρωση των εξισώσεων των μεταβολών εφαρμόζαμε τη μέθοδο Tangent Map (TM). Ο αλγόριθμος που δημιουργήσαμε εφαρμόσαμε σε τρία εξωπλανητικά συστήματα τα GJ 876, HR 8799 και TRAPPIST-1. Ο Γρήγορος Δείκτης Lyapunov (Fast Lyapunov Indicator, FLI) του συστήματος, υπολογίζεται με σκοπό την ταξινόμηση των τροχιών στο σύστημα σε τακτικές και χαοτικές. Εφαρμόζουμε επίσης το κριτήριο σταθερότητας κατά την ποσότητα Angular Momentum Deficit (AMD) ή Έλλειμμα Στροφορμής για την κατηγοριοποίηση των συστημάτων ως ευσταθή ή ασταθή κατά την ποσότητα αυτή.
iv
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The purpose of this study is the symplectic integration of the equations of motion and the variational equations for a N - body system. Extrasolar systems are rapidly discovered nowadays. Only with Kepler mission, there are 4,496 candidate exoplanets discovered and 2,330 confirmed exoplanets. Many of those exoplanets are part of multi - planet systems. The importance of that discoveries are the existence of exoplanets about Earth-size in the habitable zone and the architecture resemblance of some extrasolar systems with our Solar system. Since many of the extrasolar systems consists of at least two exoplanets the problem corresponds to a N - body simulation where the bodies are in gravitational interactions.

Chapter 1 starts with an introduction to the N-body planetary problem. We describe the Hamiltonian formalism of the system and set the system of equations of motions and the system of variational equations of the system. Then we give a short description of the tools we use for the detection of chaos and the classification of a system as long - term stable.

In Chapter 2, we analyse the algorithm of the symplectic integrator used in the simulations of this study. We explain how this method is applied on both the set of equations of motion and the variational equations of the system simultaneously. We also compare the symplectic integrator we use with Burlich - Stoer method.

Chapter 3 contains the simulations we executed. We present the results of the simulation on the three extrasolar systems we selected for various initial conditions.

Finally, in Chapter 4 we present a summary of our conclusions and analyse shortly our thoughts of future research.

At this point, I would like to thank my supervisor assosiate professor Mr. George Voyatzis. The door to Prof. Voyatzis office was always open whenever I ran into a trouble spot or had a question about my research or writing. He consistently allowed this paper to be my own work, but directed me in the right way whenever he thought I needed it.

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sister Despoina for her advise, patience and compassion and my aunt Evaggelia Ntigklidou for all those conversations we had when I was troubled. This accomplishment would not have been possible without them. Thank you.

Dimitra Skoulidou

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Chapter 1

Introduction

1.1 Kepler’s laws of planetary motion

According to Nicolaus Copernicus, the Sun is positioned near the center of the universe, motionless, with the other planets rotating around it in circular paths with epicycles and at uniform speeds. His model was published in 1543 and is called Copernican heliocentrism (Figure 1.1).

![Heliocentric model from Nicolaus Copernicus](image)

Figure 1.1: Heliocentric model from Nicolaus Copernicus

Johannes Kepler work was published between 1609 (the first and second law) and 1619 (the third law) and improved the heliocentric theory of Nicolaus Coper-
nicus, explaining how the planets’ speeds varied and using elliptical orbits. The confirmation that Kepler’s laws could apply in the Solar System as a consequence of Newton’s laws of motion and law of universal gravitation, came from Isaac Newton himself in 1687.

The Kepler’s laws of planetary motion are

1. The orbit of the planet is an ellipse with the Sun at one of the two foci. (Figure 1.2)

2. A line segment joining a planet and the Sun sweeps out equal areas during equal interval of time. (Figure 1.3)

3. The square of the orbital period \( T \) of a planet is proportional to the cube of the semi-major axis \( a \) of its orbit.

\[
T^2 \propto a^3 \quad (1.1)
\]

Figure 1.2: Kepler’s First Law: An elliptic orbit.

Figure 1.3: Kepler’s Second Law
1.1. KEPLER’S LAWS OF PLANETARY MOTION

The parameters required to uniquely identify a particular orbit are called *orbital elements* or *Keplerian elements*.

Let assume that two bodies mutually interact with gravitational forces. When they are observed from an inertial frame, the two bodies have distinct trajectories, each of these trajectories has its focus at the common center of mass. When they are observed from a non-inertial frame centred on one of the bodies, called primary body, we can describe the trajectory of the other body, called secondary body, by using the Keplerian elements below:

- **Semi major axis** ($a$) is the sum of the periapsis and apoapsis distances divided by two. For circular orbits, the semi major axis is the distance between the centers of the bodies, not the distance of the bodies from the center of mass.

- **Eccentricity** ($e$) defines the shape of the ellipse, describing how much it is elongated compared to a circle.

- **Inclination** ($i$): vertical angle of the ellipse with respect to the reference plane, measured at the ascending node (where the orbit passes upward through the reference plane, the green angle $i$ in figure 1.4). The vertical angle is measured perpendicular to line of intersection between orbital plane and reference plane. Any three points on an ellipse will define the ellipse orbital plane. The plane and the ellipse are both two-dimensional objects defined in three-dimensional space.

- **Longitude of the ascending node** ($\Omega$): horizontally orients the ascending node of the ellipse (where the orbit passes upward through the reference plane) with respect to the reference frame’s vernal point (the green angle $\Omega$ in figure 1.4).
• Argument of periapsis ($\omega$) defines the orientation of the ellipse in the orbital plane, as an angle measured from the ascending node to the periapsis (the closest point the satellite object comes to the primary object around which it orbits, the blue angle $\omega$ in figure 1.4).

• True anomaly ($v$) at epoch defines the position of the orbiting body along the ellipse at a specific time (the ”epoch”).

Some more orbital elements that we are going to use:

• Mean Anomaly ($M$) gives an angular distance from the pericentre at arbitrary time. Mean anomaly does not measure an angle between any physical objects. Is defined as

$$M = \frac{2\pi}{T}(t - \tau)$$  \hspace{1cm} (1.2)

where $\tau$ is the time at which the body is at the pericenter.

• Longitude of perihelion ($\varpi$) is the sum of the longitude of the ascending node $\Omega$, and the argument of perihelion $\omega$.

$$\varpi = \Omega + \omega$$  \hspace{1cm} (1.3)

### 1.2 Equations of motions

The main topic of this section is the presentation of the equations of motion of planetary system.

To begin with, by dynamical system we refer to a physical and/or mathematical system that evolves in what we usually call time $t$ which is the independent real variable for our model. The state of such a system is defined by the values of a set of $n$ real variables $x_1, x_2, \ldots, x_n$ and the evolution of that state is derived from a well-defined rule with respect to time. The dimension of the system is the number $l$ of state variables. We represent the state of the system by using the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The $n$-dimensional space $\mathbb{R}^n$ is called phase space of the system and a state $\mathbf{x}$ at a particular value of time represents a point in that space. Orbit of the dynamical system is defined as a set of states $\mathbf{x}(t)$.

Dynamical systems are classified mainly in two types:

1. **Continuous dynamical system** described by differential equations of the form

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t)$$  \hspace{1cm} (1.4)

with $\mathbf{f}$ being a set of $n$ functions that form a vector field. Time $t$ in this case is a continuous variable. That means that time can take any real value.
2. **Discrete dynamical system or maps**, described by differential equations of the form

\[ x_{i+1} = f(x_i, t) \]  

(1.5)

with \( f \) being a set of \( n \) functions. The state of the system \( x_i \) describe the vector \( x \) at a discrete time \( t = i \) (integer).

For a certain time \( t_0 \) we refer to the state of the system \( x_0 = (x_{10}, x_{20}, \ldots, x_{n0}) \) as the **initial state** of the system. The initial state can be represented by any point of the phase space \( R^n \) and the set of equations applies in any initial state. If the evolution of the system is always the same using a certain initial condition, then this system is called **deterministic**. On the other hand, if there is a level of randomness in the evolution of the dynamical system then this system is called **stochastic**.

A deterministic dynamical system is **autonomous** when the rule of the system does not include time as a variable, on the contrary to a **non-autonomous** dynamical system where \( t \) appears explicitly in the vector field of the system. For the autonomous dynamical systems, the evolution of the system in time is irrelevant to the choice of the initial time value \( t_0 \).

In our study, we focus on a certain type of dynamical systems called **Hamiltonian systems**. An autonomous Hamiltonian system of \( n \) degrees of freedom is described by \( 2n \) variables \( p_i, q_i, i = 1, \ldots, n \) and the differential equations

\[
\dot{q}_i = \frac{\partial H}{\partial p_i} \\
\dot{p}_i = -\frac{\partial H}{\partial q_i}
\]  

(1.6)

with \( q_i \) and \( p_i \), the generalized coordinates and conjugate momenta respectively and

\[ H = H(q_1, q_2, \ldots, q_n, p_1, p_1, \ldots, p_n, t) \]  

(1.7)

the Hamiltonian function of the system.

The vector field of a Hamiltonian system is

\[
f = \left( \frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, \ldots, \frac{\partial H}{\partial p_n}, \frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}, \ldots, \frac{\partial H}{\partial q_n} \right)
\]  

(1.8)

Since the divergence of the vector field \( \text{div} f \) is equal to zero, the Hamiltonian system **conservative**. This means that the Hamiltonian function of the system remains a constant through the evolution of the system in time. For Hamiltonian equation (1.6), the derivative with respect to time is

\[
\frac{dH}{dt} = 0 \Rightarrow H = \text{const}
\]  

(1.9)

thus the Hamiltonian of the system is a constance and it depends only from the initial state of the system for time \( t_0 \).
CHAPTER 1. INTRODUCTION

1.2.1 Equations of motion of planets

The form of a dynamical system of \((N + 1)\) bodies in gravitational interaction is presented below. The masses of the bodies will be noted as \(m_i\) with \(i = 0, 1, 2, \ldots, N\). Each body in this system has three degrees of freedom, thus the planetary system is described by \(6(N + 1)\) variables. The state of the system is defined in a Cartesian \(R^{3(N + 1)}\) frame by the vector

\[
\mathbf{x}(t) = (\mathbf{q}(t), \mathbf{p}(t))
\]  

(1.10)

where \(\mathbf{q}(t) = \{q_{ik} | i = 0, 1, \ldots, N, k = 1, 2, 3\}\) and \(\mathbf{p}(t) = \{p_{ik} | i = 0, 1, \ldots, N, k = 1, 2, 3\}\). The index \(k = 1, 2, 3\) indicates the Cartesian variables \(x, y, z\) respectively. The Hamiltonian of the system is written as:

\[
H = T(\mathbf{p}) + V(\mathbf{q}) = \frac{1}{2} \sum_{i=0}^{N} \frac{||\mathbf{p}_i||^2}{m_i} - G \sum_{i,j=0, i \neq j}^{N} \frac{m_i m_j}{||\mathbf{r}_{ij}||}
\]

(1.11)

where \(G\) is the constant of gravitation and

\[
||\mathbf{r}_{ij}|| = ||\mathbf{q}_i - \mathbf{q}_j||
\]

(1.12)

with \(\mathbf{q}_i = \{q_{i1}, q_{i2}, q_{i3}\}\) and \(\mathbf{q}_j = \{q_{j1}, q_{j2}, q_{j3}\}\).

Then the Hamiltonian equations (1.6) take the form

\[
\dot{\mathbf{x}} = \begin{cases} 
\dot{\mathbf{q}} = \mathbf{p}, & \mathbf{p} = -\frac{\partial V(\mathbf{q})}{\partial \mathbf{q}} 
\end{cases}
\]

(1.13)

For this system the center of mass is defined by the equation

\[
\mathbf{x}_{BC} = \frac{1}{M} \sum_{i=0}^{N} m_i \mathbf{x}_i \Rightarrow \left\{ \begin{array}{ll}
\mathbf{q}_{CM} = \frac{1}{M} \sum_{i=0}^{N} m_i \mathbf{q}_i \\
\mathbf{p}_{CM} = \frac{1}{M} \sum_{i=0}^{N} m_i \mathbf{p}_i
\end{array} \right.
\]

(1.14)

where \(M = \sum_{i=0}^{N} m_i\) is the total mass of the system.

Solving (1.14) in respect to the coordinates of a certain body \(i, 0 \leq i \leq N\), it gives

\[
\mathbf{q}_i = \frac{M}{m_i} \mathbf{q}_{BC} - \frac{1}{m_i} \sum_{j=0, j \neq i}^{N} m_j \mathbf{q}_j
\]

(1.15)

In an inertial reference frame where \(O[0, 0, 0]\) is the center of mass, (1.15) becomes

\[
\mathbf{q}_i = -\frac{1}{m_i} \sum_{j=0, j \neq i}^{N} m_j \mathbf{q}_j
\]

(1.16)
1.2. EQUATIONS OF MOTIONS

The derivative of (1.16) in respect with time gives the velocity of body \( i \) and is equal to

\[
\mathbf{q}_i = -\frac{1}{m_i} \sum_{j=0,j\neq i}^{N} m_j \mathbf{q}_j \Rightarrow \mathbf{p}_i = -\frac{1}{m_i} \sum_{j=0,j\neq i}^{N} m_j \mathbf{p}_j
\]  

(1.17)

The zero-body will be the star of our planetary system and its state is given by (1.16) and (1.17) for \( i = 0 \). Thus, the system of the equations of motion (1.13) can refer only for the \( N \) bodies moving around the star in a barycentric reference frame. The order of the system is reduced from \( 6(N+1) \) to \( 6N \).

By taking into consideration that \( \mathbf{p}_i = \mathbf{q}_i \) and thus \( \mathbf{p}_i = \mathbf{q}_i \), we obtain from (1.13)

\[
m_i \ddot{\mathbf{q}}_i = G \sum_{i,j=1,i\neq j}^{N} \frac{m_i m_j}{||\mathbf{r}_{ij}||^3} \mathbf{r}_{ij}
\]  

(1.18)

where \( \mathbf{r}_{ij} = \mathbf{q}_j - \mathbf{q}_i \). Summarizing the external products of \( \mathbf{q}_i \) and \( \mathbf{q}_i \) for each of (1.18), we have

\[
\sum_{i=1}^{N} m_i \mathbf{q}_i \times \ddot{\mathbf{q}}_i = G \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{m_i m_j}{||\mathbf{r}_{ij}||^3} \mathbf{q}_i \times \mathbf{r}_{ij}, \quad i \neq j
\]  

(1.19)

However,

\[
\mathbf{q}_i \times \mathbf{r}_{ij} = \mathbf{q}_i \times (\mathbf{q}_j - \mathbf{q}_i) = \mathbf{q}_i \times \mathbf{q}_j
\]

and

\[
\mathbf{q}_j \times \mathbf{r}_{ij} = \mathbf{q}_j \times \mathbf{q}_i = -\mathbf{q}_i \times \mathbf{q}_j
\]

Thus, equation (1.19) becomes

\[
\sum_{i=1}^{N} m_i \mathbf{q}_i \times \ddot{\mathbf{q}}_i = 0
\]  

(1.20)

and after an integration we have

\[
\sum_{i=1}^{N} m_i \mathbf{q}_i \times \dot{\mathbf{q}}_i = C \Rightarrow \sum_{i=1}^{N} \mathbf{q}_i \times \mathbf{p}_i = C
\]  

(1.21)

This leads us to say that the total angular momentum of the system, given by

\[
\mathbf{L} = \sum_{i=1}^{N} \mathbf{q}_i \times \mathbf{p}_i
\]  

(1.22)
is a constant of the planetary system.

If we use the orbital elements \((a_i, e_i, i_i, \omega_i, \Omega_i)\), the norm of the angular momentum with respect to the reference plane will be equal to

\[
G = \sum_{i=1}^{N} \Lambda_i \sqrt{1 - e_i^2} \cos i_i
\]

(1.23)

with \(\Lambda_i = m_i \sqrt{\mu a_i}\) and \(\mu = G m_0\).

### 1.3 Variational Equations

Let us assume that a disturbed state \(\tilde{x}(t)\) in the neighbourhood of the state \(x(t)\), has the form

\[
\tilde{x}(t) = x(t) + w(t) \Rightarrow \begin{cases}
\dot{q}(t) = q(t) + \delta q(t) \\
\dot{p}(t) = p(t) + \delta p(t)
\end{cases}
\]

(1.24)

where

\[
w(t) = (\delta q(t), \delta p(t))
\]

(1.25)

being the variation vector and the variations \(\delta q(t) = \{\delta q_{ik} | i = 0, 1, \ldots N, k = 1, 2, 3\}\), \(\delta p(t) = \{\delta p_{ik} | i = 0, 1, \ldots N, k = 1, 2, 3\}\) are small at least for short time intervals.

The state (1.24) should verifies the equations (1.6). By replacing (1.24) and developing the equations (1.6) around the certain state \(x(t)\), skipping terms of second and higher orders with respect to variations, we get the system

\[
\dot{w} = \begin{cases}
\delta \dot{q} &= \delta p \\
\delta \dot{p} &= -D^2 H(q(t)) \delta q
\end{cases}
\]

(1.26)

with \(D^2 H\) being the Hessian matrix of hamiltonian (1.11)

\[
D^2 H(q(t)) = \frac{\partial^2 H(q, p)}{\partial q_k \partial q_l}
\]

(1.27)

where \(k, l\) refer to the generalized coordinates of the system, here \(k, l = 1, 2, \ldots, 3N\).

Note that (1.27) refers directly to the variables of the whole system, thus the indices \(k\) and \(l\) refers to the length \(3N\) of vectors \(q(t)\).

The ‘tangent dynamics Hamiltonian’ (THD) is represented by the time depended Hamiltonian function

\[
H_V(\delta q, \delta p; t) = \frac{1}{2} \sum_{k=1}^{3N} \delta p_k^2 + \frac{1}{2} \sum_{k, l=1}^{3N} D^2 H(q(t))_{kl} \delta q_k \delta q_l
\]

(1.28)
1.3. VARIATIONAL EQUATIONS

The equation (1.28) represents the tangent dynamics of Hamiltonian (1.11).

In the planetary system, since the state of the zero-body is given by (1.16) and (1.17) in respect to the coordinates of the other bodies, the system of the variational equations (1.26) that occurs, refers to the N-bodies moving around it. The order of this system will also be $6N$. The deviation vector $w_i(t)$ of a certain body $i$ of the planetary system, according to (1.26) and (1.11), is given by

$$ w_i = \left\{ \begin{array}{ll} \delta q_i = & \delta p_i = -D^2_V(q_i(t)) \delta q_i \\ \end{array} \right. $$

(1.29)

where $\delta q_i = \{\delta q_{i1}, \delta q_{i2}, \delta q_{i3}\}$, $\delta p_i = \{\delta p_{i1}, \delta p_{i2}, \delta p_{i3}\}$ and $q_i = \{q_{i1}, q_{i2}, q_{i3}\}$ with

$$ D^2_V(q_i(t)) = \begin{pmatrix} \frac{\partial^2 V(q_i)}{\partial q_{i1}^2} & \frac{\partial^2 V(q_i)}{\partial q_{i1} \partial q_{i2}} & \frac{\partial^2 V(q_i)}{\partial q_{i1} \partial q_{i3}} \\ \frac{\partial^2 V(q_i)}{\partial q_{i2} \partial q_{i1}} & \frac{\partial^2 V(q_i)}{\partial q_{i2}^2} & \frac{\partial^2 V(q_i)}{\partial q_{i2} \partial q_{i3}} \\ \frac{\partial^2 V(q_i)}{\partial q_{i3} \partial q_{i1}} & \frac{\partial^2 V(q_i)}{\partial q_{i3} \partial q_{i2}} & \frac{\partial^2 V(q_i)}{\partial q_{i3}^2} \end{pmatrix} $$

(1.30)

Thus,

$$ D^2_V(q_i) \delta(q_i) = \begin{pmatrix} \frac{\partial^2 V(q_i)}{\partial q_{i1}^2} \delta(q_{i1}) + \frac{\partial^2 V(q_i)}{\partial q_{i1} \partial q_{i2}} \delta(q_{i2}) + \frac{\partial^2 V(q_i)}{\partial q_{i1} \partial q_{i3}} \delta(q_{i3}) \\ \frac{\partial^2 V(q_i)}{\partial q_{i2} \partial q_{i1}} \delta(q_{i1}) + \frac{\partial^2 V(q_i)}{\partial q_{i2}^2} \delta(q_{i2}) + \frac{\partial^2 V(q_i)}{\partial q_{i2} \partial q_{i3}} \delta(q_{i3}) \\ \frac{\partial^2 V(q_i)}{\partial q_{i3} \partial q_{i1}} \delta(q_{i1}) + \frac{\partial^2 V(q_i)}{\partial q_{i3} \partial q_{i2}} \delta(q_{i2}) + \frac{\partial^2 V(q_i)}{\partial q_{i3}^2} \delta(q_{i3}) \end{pmatrix} $$

(1.31)

To calculate the matrix $D^2_V(q_i(t))$ we have to take into consideration that the thesis of the zero-body is a linear combination of the thesis of the N-bodies.

To be more specific the second derivative of $V(q_i)$ in respect of the $i$-body’s coordinates, $\{q_{i1}, q_{i2}, q_{i3}\}$, will be given by

$$ \frac{\partial^2 V}{\partial q_{i1} \partial q_{ik}} = \begin{cases} \sum_{j=1,j\neq i}^N \left[ -m_i m_j \frac{r_{ij}^2}{m_0} + 3(q_{jk} - q_{ik})^2 \right] \\ -m_0 m_i \left( 1 + \frac{m_i}{m_0} \right) \frac{r_{ij}^2}{r_{0i}^2} + 3(q_{jk} - q_{0k})^2, \quad k = l \\ \sum_{j=1,j\neq i}^N \left[ -m_i m_j \frac{3(q_{jk} - q_{0k})(q_{ji} - q_{0i})}{r_{0i}^2} \right] \\ -m_0 m_i \left( 1 + \frac{m_i}{m_0} \right) \frac{3(q_{jk} - q_{0k})(q_{ji} - q_{0i})}{r_{0i}^2}, \quad k \neq l \end{cases} $$

(1.32)

where $k, l = 1, 2, 3$. 

...
1.4 Fast Lyapunov Indicator

A mathematical way to define *chaos* is given by Devaney in [4] and presented below.

**Definition 1.** Let $V$ be a set and $f : V \rightarrow V$ a map on this set. We say that $f$ is *chaotic* on $V$ if

i. $f$ has sensitive dependence on initial conditions.
ii. $f$ is topologically transitive.
iii. periodic points are dense in $V$.

From the definition above, we understand that a chaotic system is unpredictable due to the sensitive dependence on initial conditions. Furthermore, it cannot be partitioned into two non-interacting subsystems because of the topological transitivity. Finally, the density of the periodic points leads to regularity.

The first hypothesis of the definition of chaos is the most commonly used to describe the phenomenon in physics. To explain more about the sensitive dependence on initial conditions, we give the following definition ([17]).

**Definition 2.** $f : V \rightarrow V$ has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for any $x \in V$ and any neighbourhood $\Delta$ of $x$, there exist $y \in \Delta$ and $n \geq 0$, such that $|f^n(x) - f^n(y)| > \delta$, where $f^n$ denotes $n$ successive applications of $f$.

In other words, there exist points arbitrarily close to $x$ which eventually separate from $x$ by at least $\delta$ under iterations of $f$.

As multi-planet systems are to be examined in this work, it is necessary to detect chaotic behaviour of the motion of the planets. We mention that in Hamiltonian systems there exist both regular and chaotic orbits in phase space. One way to detect if an orbit is chaotic is to calculate the *Lyapunov Characteristic Exponent* or the *Fast Lyapunov Indicator* of the system. Both of these tools are presented below.

The *Lyapunov Characteristic Exponents* (LCEs) were once introduced by Lyapunov when studying the stability of non-stationary solutions of ordinary differential equations [12]. The LCEs give the rate of exponential divergence from perturbed initial conditions. In case of chaotic orbits at least one LCE is positive, whereas for regular orbits all LCEs are zero. The computation of the maximum LCE (mLCE) $\chi_1$ is sufficient for determining whether an orbit is chaotic or not. Therefore, if $\chi_1 > 0$, the orbit is chaotic.

Let us consider, $w(0)$ being the deviation vector from the initial state $x(0)$ of the system and $w(t)$ being the deviation vector in time evolution, according to
variational equations (1.26). The characteristic exponent of order 1, $\chi_1$ is given by the Oseledec Theorem [14]

$$\chi_1 = \lim_{t \to \infty} \frac{1}{t} \log \frac{||w(t)||}{||w(0)||}$$  \hspace{1cm} (1.33)

In a chaotic region, the value of LCE $\chi_1$ is independent from the initial vector $w(0)$. Therefore, in our computations we consider $w(0)$ in way that $||w(0)|| = 1$. The value that LCE tends to, can also be used to characterize how strong is chaos.

Since the computation of deviation vectors contains a lot of information even on short integration times, Fast Lyapunov Indicator (FLI) is a tool introduced by C. Froeschle et al. [6] and relies on that idea. FLI is closely related to the computation of Lyapunov Exponent. At a certain time $t$, FLI is given by

$$FLI = \log \frac{||w(t)||}{||w(0)||}$$  \hspace{1cm} (1.34)

If the rate of increasing is slow as $t \to \infty$ then the system is considered to be regular. On the contrary, if the system is chaotic, then the value of FLI increases exponentially in time.

### 1.5 Angular Momentum Deficit (AMD)

Angular Momentum Deficit (AMD) was introduced by Laskar ([9], [10], [11]). In 1997, Laskar [9] showed that large scale chaos is responsible for the spacing of the inner planets of the Solar System and the conservation of their AMD is the only limitation that keeps them from wander and thus long term stability of the system is achieved. In 2000, a simplified model for the evolution and formation of any planetary system was introduced in [10]. The model shows that the organization of planetary systems can derive from the knowledge of their initial mass distribution. The semi major axis, the masses and the AMD of the system are the base to form a stability criterion that leads to the classification of the planetary systems [11].

The AMD is defined as the difference between the norm of the angular momentum of a coplanar and circular system with the same semi major axis values and the norm of the angular momentum (1.23) of the system. Thus,

$$C = \sum_{i=1}^{N} \Lambda_i \left( 1 - \sqrt{1 - e_i^2 \cos i_i} \right)$$  \hspace{1cm} (1.35)

A loss of AMD for the planetary system is obtained when the architecture of the system is modified. Such a modification happens when a planet is ejected from the
system or collide with another planet or the star. The reduction of AMD during collisions has as a result the stabilisation of the averaged system. A planetary system is characterised as AMD-stable when the total AMD amount cannot allow planetary collisions to happen.

If the total AMD $C$ of the system is absorbed by two consecutive planets and their orbits cannot intersect, then the condition of AMD-stability is accomplished. When the orbits do not intersect, a collision can happen when the perihelion of the one planet is the aphelion of the other. The limit condition of collisions is obtained in the planar case. Thus,

$$a(1 + e) = a'(1 - e')$$

(1.36)

and from the definition of AMD (1.35) for the system of the two planets we obtain

$$m\sqrt{\mu a}(1 - \sqrt{1 - e^2}) + m'\sqrt{\mu a'}(1 - \sqrt{1 - e'^2}) = C$$

(1.37)

We mark as $(m, a, e)$ the parameters of the inner planet and $(m', a', e')$ those of the outer planet. Thus, $a \leq a'$. If $\alpha = a/a'$ and $\gamma = m/m'$, the set of collisional conditions becomes

$$\mathcal{D}(e, e') = \alpha e + e' - 1 + \alpha = 0$$

(1.38)

and

$$\mathcal{C}(e, e') = \gamma \sqrt{\alpha}(1 - \sqrt{1 - e^2}) + (1 - \sqrt{1 - e'^2}) = C/\Lambda'$$

(1.39)

where $\Lambda' = m'\sqrt{\mu a'}$ and $e, e' \in [0, 1]$. The $\mathcal{C}(e, e')$ is called relative AMD.

The goal is to find the minimum value of the relative AMD (1.39) for which the condition (1.38) is satisfied. The eccentricity $e'$ can be eliminated from (1.38) by using the Lagrange multipliers condition. In any case the Lagrange multipliers condition is

$$\lambda \nabla \mathcal{D}(e, e') = \nabla \mathcal{C}(e, e') \Rightarrow \frac{\sqrt{1 - e'^2}}{e'} = \frac{\sqrt{\alpha}}{\gamma} \frac{\sqrt{1 - e^2}}{e}$$

(1.40)

thus, from (1.38) and (1.40), we obtain

$$F(e, \alpha, \gamma) = \alpha e + \frac{\gamma e}{\sqrt{\alpha(1 - e^2) + \gamma^2 e^2}} - 1 + \alpha = 0$$

(1.41)

In [11] it is proven that equation (1.41) always has a single solution for a given pair of $(\alpha, \gamma)$, called critical eccentricity $e_C$, where $e_C \in [0, 1]$. The $e_C$ is the critical eccentricity for the inner planet. The critical eccentricity of the outer planet $e'_C$ is calculated from (1.38). The critical value of relative AMD for a given pair of planets is obtained by (1.39), thus

$$C_C(\alpha, \gamma) = \mathcal{C}(e_C, e'_C)$$

(1.42)
1.5. ANGULAR MOMENTUM DEFICIT (AMD)

and it is unique. Note that the critical value of AMD depends only on \((\alpha, \gamma)\). It is proven by Laskar ([10], [11]) that the critical value of AMD increases with \(\gamma\) and decreases with \(\alpha\).

The system of two planets is AMD-stable if and only if the total AMD of the planetary system \(C(1.35)\) is smaller than \(\Lambda'C_C(\alpha, \gamma)\). The \(AMD\) - stability coefficient is obtained by

\[
\beta = \frac{C}{\Lambda'C_C(\alpha, \gamma)}
\]  

(1.43)

Thus, if \(\beta < 1\) collision of a given pair of planets is not possible.

For the pair of the star and the innermost planet \(\alpha\) is considered to be zero and thus \(C_C = 1\). The AMD-stability coefficient for this particular pair is \(\beta_S = C/\Lambda\) where \(\Lambda\) refers to the innermost planet.

To characterize a planetary system as AMD - stable, collisions between planets must be impossible because of AMD dynamics. Based on AMD stability criterion, a system can be characterized as:

- **Strong AMD - stable system**, when all the AMD - stability coefficients \(\beta\) calculated for all the pairs of the contiguous planets are \(\beta < 1\) and the AMD - stability coefficient of the star and the innermost planet is also \(\beta_S < 1\).

- **Weak AMD - stable system**, when all the AMD - stability coefficients \(\beta\) calculated for all the pairs of the contiguous planets there are \(\beta < 1\), whereas for the AMD - stability coefficient of the star and the innermost planet there is \(\beta_S > 1\).

- **AMD - unstable system**, when the system must have at least one unstable planet pair, that means at least one \(\beta > 1\).

In the second case, the weak AMD - stable system, the eccentricity of the innermost planet can be increased up to 1 and the planet can collide with the star. The remaining system will not be affected by the destruction of that planet.
CHAPTER 1. INTRODUCTION
Chapter 2

Numerical Integration

2.1 Symplectic Integrator

A symplectic integrator is a numerical integration scheme for Hamiltonian systems. In this chapter we are going to discuss how we can integrate the equations of motion (1.13) by using the symplectic scheme of 6th order presented by Yoshida in [20].

The symplectic two-form \( dp \times dq \) (Louville Theorem),

\[
dp \times dq = dp' \times dq' \tag{2.1}
\]

and the total energy of the system

\[
H(q, p) = H(q', p') = const \tag{2.2}
\]

should be preserved during the study of a Hamiltonian system. According to Ge and Marsden [7] there is no energy conserving methods, like Runge-Kutta, applied to non integrable Hamiltonian systems that fulfils the two conditions above. However, symplectic integrators is based on the symplectic form and the energy error is bounded.

Let us define the Poisson bracket for two functions \( F(q, p) \) and \( G(q, p) \) is

\[
\{ F, G \} = \sum_{k=1}^{3N} \left( \frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right) = L_{G}F \tag{2.3}
\]

Then, the equations of motion (1.13) is written as

\[
\dot{x} = \{ x, H \} = L_{H}x \tag{2.4}
\]

and the exact time evolution of (2.4) for time step \( \tau \) is given by

\[
x(t + \tau) = e^{\tau L_{H}}x(t) = e^{\tau(A+B)}x(t) \tag{2.5}
\]
where $A = L_T$ and $B = L_V$ are two integrable parts of the Hamiltonian and $L_H = L_T + L_V$, with

$$L_T(p) = \frac{1}{2} \sum_{i=0}^{N} \frac{||p_i||^2}{m_i} \quad \text{and} \quad L_V(q) = V(q) \quad (2.6)$$

The exponential operator $e^{\tau(A+B)}$ can be approached by an integrator of $k$ steps produced by the composition of the products of $e^{c_i\tau A}$ and $e^{d_i\tau B}$, \( i = 1, 2, \ldots, k \). The constants $c_i$ and $d_i$ of the elementary symplectic maps are real numbers. A symplectic integrator of $nth$ order is obtained by the action of the composition of the products of $e^{c_i\tau A}$ and $e^{d_i\tau B}$ on $x(t)$, thus

$$e^{\tau(A+B)} = \left( \prod_{i=1}^{k} e^{c_i\tau A} e^{d_i\tau B} \right) + O(\tau^{n+1}) \quad (2.7)$$

and (2.5) is equal to

$$x(t + \tau) = \left( \prod_{i=1}^{k} e^{c_i\tau A} e^{d_i\tau B} \right) x(t) \quad (2.8)$$

The two elementary symplectic maps involving the set of initial conditions $(q, p)$ at time $t$ to their final values $(q', p')$ at time $t + \tau$ are

$$e^{c_i\tau A} : \begin{cases} q' = q + c_i p \tau \\ p' = p \end{cases} \quad (2.9)$$

and

$$e^{d_i\tau B} : \begin{cases} q' = q \\ p' = p - d_i \frac{\partial V(q)}{\partial q} \tau \end{cases} \quad (2.10)$$

For the calculation of the constants $c_i$ and $d_i$, we solve the set of non-linear algebraic equations obtained by expanding the left hand side of (2.8) in powers of $\tau$ and equating the coefficient of the equal powers of $\tau$ up to the order $\tau^n$.

To construct a 2nd order integrator we use the Baker - Campbell - Hausdorff (BCH) [5] formula and the leap-frog method

$$S_{2nd} = e^{\frac{\tau}{2} A} e^{\tau B} e^{\frac{\tau}{2} A} \quad (2.11)$$

Since the symplectic integrator is symmetric, Yoshida [20] showed that a higher order symplectic integrator can be constructed from lower order integrators. More specifically if a symmetric integrator of order $2n$, $S_{2n}(\tau)$ is already known, a $(2n + 2)$th order integrator is obtained by the product

$$S_{2n+2}(\tau) = S_{2n}(z_1 \tau) S_{2n}(z_0 \tau) S_{2n}(z_1 \tau) \quad (2.12)$$
2.1. SYMPLECTIC INTEGRATOR

where \( z_0 \) and \( z_1 \) must satisfy

\[
    z_0 + 2z_1 = 1, \quad z_0^{2n+1} + 2z_1^{2n+1} = 0
\]

(2.13)

or

\[
    z_0 = -\frac{2^{1/(2n+1)}}{2 - 2^{1/(2n+1)}}, \quad z_1 = \frac{1}{2 - 2^{1/(2n+1)}}
\]

(2.14)

A 4th order symplectic integrator can be produced by the combination of 2nd order integrators, thus according to the BCH formula

\[
    S_{4th}(\tau) = S_{2nd}(x_1\tau)S_{2nd}(x_0\tau)S_{2nd}(x_1\tau)
\]

(2.15)

where

\[
    x_0 + 2x_1 = 1, \quad x_0^3 + 2x_1^3 = 0
\]

(2.16)

or

\[
    x_0 = -\frac{2^{1/3}}{2 - 2^{1/3}}, \quad x_1 = \frac{1}{2 - 2^{1/3}}
\]

(2.17)

By using (2.11), (2.15) becomes

\[
    S_{4th}(\tau) = e^{x_1\tau A}e^{x_1\tau B}e^{x_1\tau A}
\]

\[
    e^{x_0\tau A}e^{x_0\tau B}e^{x_0\tau A}
\]

\[
    e^{x_1\tau A}e^{x_1\tau A}e^{x_1\tau A}
\]

(2.18)

If the successive maps of A are joined together then the symplectic map of 4th order becomes a product of seven maps of A and B.

\[
    S_{4th}(\tau) = e^{c_1\tau A}e^{d_1\tau B}e^{c_2\tau A}e^{d_2\tau B}e^{c_3\tau A}e^{d_3\tau B}e^{c_4\tau A}
\]

(2.19)

where

\[
    c_1 = c_4 = \frac{x_1}{2}, \quad c_2 = c_3 = \frac{x_0 + x_1}{2}
\]

\[
    d_1 = d_4 = x_1, \quad d_2 = x_0
\]

(2.20)

Thus, a 4th order symplectic integrator needs seven sub-steps to fulfil one time step. Both Forest and Ruth (1990) \[3\] and Candy and Rosmus (1991) \[2\] came to the same calculations for the coefficients (2.20) of the 4th order symplectic integrator.

The 6th order integrator can be written as a product of three 4th order integrators

\[
    S_{6th}(\tau) = S_{4th}(w_1\tau)S_{4th}(w_0\tau)S_{4th}(w_1\tau)
\]

(2.21)

or nine 2nd order integrators

\[
    S_{6th}(\tau) = S_{2nd}(x_1w_1\tau)S_{2nd}(x_0w_1\tau)S_{2nd}(x_1w_1\tau)
\]

\[
    \times S_{2nd}(x_1w_0\tau)S_{2nd}(x_0w_0\tau)S_{2nd}(x_1w_0\tau)
\]

\[
    \times S_{2nd}(x_1w_1\tau)S_{2nd}(x_0w_1\tau)S_{2nd}(x_1w_1\tau)
\]

(2.22)
To reduce the total substeps needed for 6th order symplectic integrator, Yoshida [20] simplified the procedure by calculating three solutions of three coefficients, and then calculating the constants $c_i$ and $d_i$ by using those. The sub-steps are reduced to fifteen from twenty seven.

To be more specific

\[
\begin{align*}
  w_0 &= 1 - 2(w_1 + w_2 + w_3) \\
  w_1 &= -1.177679984178872038597 \\
  w_2 &= 0.23557321335935624273538 \\
  w_3 &= 0.78451361047756185129742
\end{align*}
\] (2.23)

and $c_i$ and $d_i$ will be equal to

\[
\begin{align*}
  c_1 &= c_8 = \frac{w_3}{2} & d_1 &= d_7 = w_3 \\
  c_2 &= c_7 = \frac{w_3 + w_2}{2} & d_2 &= d_6 = w_2 \\
  c_3 &= c_6 = \frac{w_2 + w_1}{2} & d_3 &= d_5 = w_1 \\
  c_4 &= c_5 = \frac{w_1 + w_0}{2} & d_4 &= w_0
\end{align*}
\] (2.24)

The analytical steps of the 2nd, the 4th and the 6th order symplectic integrators are shown below

\[
S_{2nd}(t + \tau) : \begin{cases} 
q_{t+\tau/2} = q_t + \frac{\tau}{2}p_t \\
p_{t+\tau} = p_t - \tau \frac{\partial V}{\partial q}|_{q_{t+\tau/2}} \\
q_{t+\tau} = q_{t+\tau/2} + \frac{\tau}{2}p_{t+\tau}
\end{cases}
\] (2.25)

\[
S_{4th}(t + \tau) : \begin{cases} 
q_{t+c_1\tau} = q_t + c_1\tau p_t \\
p_{t+d_1\tau} = p_t - d_1\tau \frac{\partial V}{\partial q}|_{q_{t+c_1\tau}} \\
q_{t+(c_1+c_2)\tau} = q_{t+c_1\tau} + c_2\tau p_{t+d_1\tau} \\
p_{t+(d_1+d_2)\tau} = p_{t+d_1t} - d_2\tau \frac{\partial V}{\partial q}|_{q_{t+(c_1+c_2)\tau}} \\
q_{t+(c_1+c_2+c_3)\tau} = q_{t+(c_1+c_2)\tau} + c_3\tau p_{t+(d_1+d_2)\tau} \\
p_{t+\tau} = p_{t+(d_1+d_2)\tau} - d_3\tau \frac{\partial V}{\partial q}|_{q_{t+(c_1+c_2+c_3)\tau}} \\
q_{t+\tau} = q_{t+(c_1+c_2+c_3)\tau} + c_4\tau p_{t+\tau}
\end{cases}
\] (2.26)

For the 4th order integrator scheme the coefficients are given by (2.20).
For the 6th order integrator scheme the coefficients are given by (2.24).
2.2 Integration Method of Variational Equations

The equations of motion (1.13) and the variational equations (1.29) of a planetary system have to be integrated simultaneously due to the dependence of matrix $D^2_V(t)$ on a particular orbit (1.10). After obtaining the values of $x(t)$ for each planet, where $t$ is a multiple of the integration step $\tau$, by using any symplectic or non-symplectic scheme, the solution of the variational equations can be calculated by using various methods. Skokos and Gerlach in [18] present three methods

1. the tangent dynamics Hamiltonian with piecewise constant coefficients (TDHcc)
2. the integration of the tangent dynamics Hamiltonian in an extended phase space (TDHeps)
3. the tangent map method (TM)

In this study the variational equations are solved by using the third method in row, the TM method.

First of all we consider a unified vector $u$ formed by both (1.10) and (2.25), thus

$$\mathbf{u} = (\mathbf{q}, \mathbf{p}, \delta \mathbf{q}, \delta \mathbf{p})$$

The set of differential equations will become

$$\dot{\mathbf{u}} = \begin{cases} \dot{\mathbf{q}} = & \mathbf{p} \\ \dot{\mathbf{p}} = & -\frac{\partial V(p)}{\partial \mathbf{p}} \\ \delta \dot{\mathbf{q}} = & \delta \mathbf{p} \\ \delta \dot{\mathbf{p}} = & -D^2_V(\mathbf{q})\delta \mathbf{q} \end{cases}$$

or

$$\dot{\mathbf{u}} = L_{HV}\mathbf{u}$$

where $L_{HV}$ is the differential operator for the whole system. In proportion to (2.5), the solution of the unified system can be written as

$$\mathbf{u}(t + \tau) = e^{\tau L_{HV}} \mathbf{u}(t)$$

It should be noted that there cannot be such a generalized Hamiltonian function with corresponding Hamilton’s equations of motions the equations (2.29). As already shown in the previous chapter the integration of Hamilton’s equations of motion (1.13) can split into steps produced by the composition of the products of $e^{c_i \tau A}$ and $e^{d_i \tau B}$. During the intermediate steps for the Hamiltonian differential operator $A$, the differential variational equations of the unified system are

$$\begin{align*}
\delta \dot{\mathbf{q}} &= \delta \mathbf{p} \\
\delta \dot{\mathbf{p}} &= 0
\end{align*}$$
and during the intermediate steps for the Hamiltonian differential operator $B$, they are
\[
\begin{align*}
\delta \dot{q} &= 0 \\
\delta \dot{p} &= -D_V^2(q)\delta q
\end{align*}
\] (2.33)

Thus for the unified set of equations for the intermediate steps of the symplectic integration there are
\[
\dot{u} = A_V u \Rightarrow \begin{cases}
q &= p \\
\dot{p} &= 0 \\
\delta q &= \delta p \\
\delta \dot{p} &= 0
\end{cases}
\] (2.34)

and
\[
\dot{u} = B_V u \Rightarrow \begin{cases}
q &= 0 \\
p &= -\frac{\partial V(q)}{\partial q} \\
\delta q &= 0 \\
\delta \dot{p} &= -D_V^2(q)\delta q
\end{cases}
\] (2.35)

with $A_V$ and $B_V$ being the corresponding differential operators of $A$ and $B$.

The two elementary symplectic maps related to the equations (2.29) of the unified system are
\[
e^{c_tA_V} : \begin{cases}
q' &= q + c_t p \\
p' &= p \\
\delta q' &= \delta q + c_t \delta p \\
\delta p' &= \delta p
\end{cases}
\] (2.36)

\[
e^{d_tB_V} : \begin{cases}
q' &= q \\
p' &= p - d_t \frac{\partial V(q)}{\partial q} \\
\delta q' &= \delta q \\
\delta p' &= \delta p - d_t D_V^2(q) \delta q
\end{cases}
\] (2.37)

The TM method can be applied to any symplectic integration scheme used to solve the Hamilton’s equations of motion (1.13) and involves the split of the Hamiltonian into two integrable parts A and B. Then the Hamilton’s equations of motion and the variational equations can simultaneously be integrated by replacing the integration maps $e^{rA}$ and $e^{rB}$ with the maps $e^{rA_V}$ and $e^{rB_V}$.

Since we consider a planetary system in the three-dimensional euclidean space, the state of each body for the unified system of equations (2.29), is described by a vector $u_i(t)$ given by (2.28) and each of the vectors $q_i$, $p_i$, $\delta q_i$ and $\delta p_i$ is defined by three elements in respect to the three coordinates of the euclidean space. In Chapter 1, sections II and III, we have already mentioned that both the equations of motion and the variational equations are to be solved only for the $N$ bodies moving around the star. The body with zero index is the star of the planetary system and its $q_0$ and $p_0$ vectors are given by (1.16) and (1.17) respectively.
For the entire N body system we have to solve N unified sets of differential equations. That means that the entire problem consists of \(12N\) differential equations. Thus

\[ q(t) = \{q_{ik}(t)|i = 1, \ldots N, k = 1, 2, 3\} \quad (2.38) \]

\[ p(t) = \{p_{ik}(t)|i = 1, \ldots N, k = 1, 2, 3\} \quad (2.39) \]

\[ \delta q(t) = \{\delta q_{ik}(t)|i = 1, \ldots N, k = 1, 2, 3\} \quad (2.40) \]

\[ \delta p(t) = \{\delta p_{ik}(t)|i = 1, \ldots N, k = 1, 2, 3\} \quad (2.41) \]

To implement the symplectic maps (2.36) and (2.37) according to the analytical steps of the 6th order integrator (2.27), it is preferable to set the variables to a specific series as shown below

\[ u(t) = \{\{q_{i1}(t), q_{i2}(t), q_{i3}(t)\}, \{\delta q_{i1}(t), \delta q_{i2}(t), \delta q_{i3}(t)\}, \{p_{i1}(t), p_{i2}(t), p_{i3}(t)\}, \{\delta p_{i1}(t), \delta p_{i2}(t), \delta p_{i3}(t)\}|i = 1, \ldots N\} \quad (2.42) \]

Thus symplectic map (2.36) affects the first 6N variables and symplectic map (2.37) the last 6N variables.

### 2.3 Symplectic Integrator Versus Bulirsch-Stoer

In this section we integrate a potential extrasolar system consisting of three planets and it shows a sufficient long term stability. Apart from the symplectic 6th order integrator of Yoshida (S\(_{6th}\)), we also use the Bulirsch-Stoer (BS) integrator. The analysis of the algorithm of the Bulirsch-Stoer method is beyond the scope of this study. However we are going to give some general information for this method.

The Bulirsch-Stoer method is used for the numerical solution of ordinary differential equations. There are three ideas combined to form this method:

- Richardson extrapolation
- the use of rational function extrapolation in Richardson - type applications
- the modified midpoint method

This method is believed to be the best known way to obtain high-accuracy solutions to ordinary differential equations with minimal computational effort according to [15].

For more information about the the Bulirsch-Stoer algorithm we used, the reader can see [15].
Table 2.1: Initial Conditions of Orbit 1. For this set $M = 1.0$. (Voyatzis [19])

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Planet 1</th>
<th>Planet 2</th>
<th>Planet 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass</td>
<td>0.001</td>
<td>0.002</td>
<td>0.0005</td>
</tr>
<tr>
<td>a (AU)</td>
<td>1.0</td>
<td>1.589</td>
<td>2.528</td>
</tr>
<tr>
<td>e</td>
<td>0.695</td>
<td>0.341</td>
<td>0.139</td>
</tr>
<tr>
<td>i (°)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$\omega$ (°)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$\Omega$ (°)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>M (°)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

The stable system is given in Table 2.1 (Voyatzis [19]). We are going to compare the errors of the total energy of the system and of the angular momentum of the system as evolving in time between the $S_{6th}$ and BS.

First we implement both methods with integration step $\tau = 0.01$. For BS integrator, the integration accuracy is $10^{-12}$ and the integration step may change in the interval $[0.0001, 0.1]$ (Case 1). In figures 2.1a and 2.1b the $S_{6th}$ method of integration is more stable and accurate than BS, although it is slower (CPU-time $S_{6th}$=195 min, CPU-time $BS$=65 min). The mean value of the logarithm of error in energy for the $S_{6th}$ is $-12.64$, while for the BS is $-9.08$. Same results can be derived for the error in angular momentum of the system, where both calculated errors are smaller.

Next we implement both methods again but with integration step $\tau = 0.001$. For BS integrator, the integration accuracy is again $10^{-12}$ and the integration step may change in the interval $[0.0001, 0.1]$ (Case 2). In Appendix A, figures A.1a and A.1b shows the results of this attempt. The conclusions remain the same, $S_{6th}$ has better results but this has same cost in computational time.

In an attempt to get the same order of error in energy for both methods, we set the integration step $\tau = 0.001$ for both methods and the integration accuracy for BS at $10^{-14}$ with the integration step interval being $[0.0001, 0.1]$ (Case 3). Figures A.2a and A.2b shows that the errors in total energy and angular momentum are closer. However, the $S_{6th}$ is still more time consuming and shows better results. The mean value of the logarithm of error in energy for the $S_{6th}$ is $-13.90$ and for the BS is $-10.82$.

Since Case 3 was not very successful, we choose integration accuracy for BS to be $10^{-15}$ with the integration step interval being $[0.0001, 0.1]$ and the integration step for both methods is equal to $\tau = 0.001$ (Case 4). Figures A.3a and A.3b show that the errors in total energy and angular momentum are much closer.
mean value of the logarithm of error in energy for the $S_{6th}$ is $-13.90$ and for the BS is $-12.90$. At the end of the integration the error in energy for BS seems to have an intense upward trend. The CPU time for the symplectic integrator is CPU-time $S_{6th}=168$ min and for Bulirsch Stoer is CPU-time $BS=98$ min.

Bulirsch Stoer is faster than the symplectic integrator of 6th order in any of the four cases we tested. However, the error in both total energy and angular momentum is significantly lower in some of those. That is the reason we chose to use the symplectic integrator of 6th order in this study, knowing of the time
2.3. SYMPLECTIC INTEGRATOR VERSUS BULIRSCH-STOER

consuming cost of its use.
Chapter 3

Extrasolar Systems

An *extrasolar planet* is a planet outside of our solar system that orbits a star. The first confirmation of an extrasolar system was made in 1995, a giant planet orbiting around 51 Pegasi. According to the Extrasolar Planets Encyclopaedia [23] there are 3,671 planets in 2,751 systems and there are 616 multi-planet systems among them (September 2017).

An extrasolar planet is designating by the name or designation of its parent star and a lower case letter. The first planet discovered in a system is getting the letter ‘b’ and later planets are given subsequent letters. When more than one planets are discovered simultaneously then the planets take their letters in order of orbital size, from the innermost to the outermost.

In this study we use the symplectic 6th order integrator for solving the unified set of differential equations (2.29) on three multi-planet systems: *Gliese 876* or *GJ 876*, *HR 8799* and *TRAPPIST-1*. For the *Gliese 876* we examine two different combinations of each of two sets of initial conditions ([1] and [16]), for the *HR 8799* we examine one set of initial conditions ([8]) and for the *TRAPPIST-1* we examine six different combinations of the initial conditions given in [23]. The FLI (1.34) is calculated for each of these extrasolar systems. In this study we are also examine the evolution of AMD - stability coefficient (1.43) of the planet pairs in time for some of those systems. Note that before we begin any integration, the masses of the star and the planets of the system are normalized so as the mass of the star will be equal to 1 and the semi-major axes $a$ of the planets are also normalized so as the distance from the star of a reference planet will be equal to 1.
3.1 GJ 876

Gliese 876 or GJ 876 is the third closest known star to the Sun confirmed to possess a planetary system. It has a distance of approximately 15 light-years away from Earth in the constellation of Aquarius. The system consists of a red dwarf with four confirmed planets orbiting the star discovered from 1998 to 2010. In 2014, two new planets where discovered GJ 876 f and GJ 876 g, but they have not been confirmed until today.

GJ 876 is an extrasolar system confirmed to be in a Laplace resonance \([\text{3}]\). It is the first extrasolar system with measured coplanarity and two of the middle planets are located in the system’s habitable zone. Unfortunately they are giant planets analogous to Jupiter.

Table 3.1: Orbital Elements for GJ 876, Set 1. For this set \(M_* = 0.334M_\odot\). (Baluev \([\text{1}]\))

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Planet d</th>
<th>Planet c</th>
<th>Planet b</th>
<th>Planet e</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass ((M_J))</td>
<td>0.0218</td>
<td>0.747</td>
<td>2.337</td>
<td>0.0482</td>
</tr>
<tr>
<td>a (AU)</td>
<td>0.02110652</td>
<td>0.131727</td>
<td>0.211018</td>
<td>0.33961</td>
</tr>
<tr>
<td>(e)</td>
<td>0.178</td>
<td>0.2498</td>
<td>0.0328</td>
<td>0.08</td>
</tr>
<tr>
<td>(i(\circ))</td>
<td>56.1</td>
<td>56.1</td>
<td>56.1</td>
<td>56.1</td>
</tr>
<tr>
<td>(\omega(\circ))</td>
<td>224.0</td>
<td>252.08</td>
<td>248.7</td>
<td>181.0</td>
</tr>
<tr>
<td>(\Omega(\circ))</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>(M(\circ))</td>
<td>133.6</td>
<td>179.01</td>
<td>92.43</td>
<td>118.3</td>
</tr>
</tbody>
</table>

By using the orbital elements from Table 3.1 (Baluev \([\text{1}]\)) we integrate the system of the three outer planets (GJ 876 c, GJ 876 b, GJ 876 e), with reference planet the GJ 876 c (GJ Set1-a) for 2.5 million years. The integration time step is \(\tau = 0.01\). The absolute error in total energy and in angular momentum are less than \(10^{-12}\) during the integration time (Figure 3.1). The FLI is increasing rapidly in time (Figure 3.2), proving the existence of chaos. The values of AMD stability coefficients at \(t = 0\) are \(\beta_S = 0.0338\), \(\beta_{eb} = 0.3009\) and \(\beta_{bw} = 4.4363\). That shows an AMD - unstable system. The evolution of the AMD stability coefficients of the system are given in figure 3.3. Both diagrams of the major - axes \(a\) of the bodies vs time and of the eccentricities of the bodies vs time are in Appendix B (Figures B.1 and B.2).

We also integrate the whole system from Table 3.1, with reference planet the GJ 876 c (GJ Set1-b) for 2.5 million years. The integration time step is \(\tau = 0.01\).
3.1. GJ 876

The diagrams of the evolution of the logarithm of the absolute error in total energy and in angular momentum of the system, the FLI and the semi major axes and eccentricities are presented in Appendix B (Figures B.3, B.4, B.5 and B.6). The absolute error in total energy is around $10^{-10}$ bigger than in GJ Set1-a. The absolute error in angular momentum is less than $10^{-12}$ during the integration time, same as in GJ Set1-a. The FLI is increasing rapidly in time, proving the existence of chaos. We also notice that the semi major axes and the eccentricities of the planets GJ 876 c, b and e have similar behaviour as in GJ Set1-a.

**Figure 3.1:** Diagrams of the logarithm of the absolute error in total energy vs time and the logarithm of the absolute error in angular momentum vs time for GJ Set1-a.

**Figure 3.2:** Diagram of the FLI vs time for GJ Set1-a.
Figure 3.3: Diagram of the AMD stability coefficients $\beta$ vs time for GJ Set1-a.

Table 3.2: Orbital Elements for GJ 876, Set 2. For this set $M_\star = 0.32 M_\odot$. (Rivera [16])

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Planet d</th>
<th>Planet c</th>
<th>Planet b</th>
<th>Planet e</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M (M_{\text{Jup}})$</td>
<td>0.0214</td>
<td>0.7142</td>
<td>2.2756</td>
<td>0.0459</td>
</tr>
<tr>
<td>a (AU)</td>
<td>0.02080665</td>
<td>0.129590</td>
<td>0.208317</td>
<td>0.3343</td>
</tr>
<tr>
<td>$e$</td>
<td>0.207</td>
<td>0.25591</td>
<td>0.0324</td>
<td>0.055</td>
</tr>
<tr>
<td>$i (^\circ)$</td>
<td>59.1</td>
<td>59.1</td>
<td>59.1</td>
<td>59.1</td>
</tr>
<tr>
<td>$\omega (^\circ)$</td>
<td>234.0</td>
<td>48.76</td>
<td>50.3</td>
<td>239.0</td>
</tr>
<tr>
<td>$\Omega (^\circ)$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$M (^\circ)$</td>
<td>355.0</td>
<td>249.59</td>
<td>325.7</td>
<td>335.0</td>
</tr>
</tbody>
</table>

By using the orbital elements from Table 3.2 (Rivera [16]), we integrate the system of the three outer planets (GJ 876 c, GJ 876 b, GJ 876 e), with reference planet the GJ 876 c (GJ Set2-a). The integration time step is $\tau = 0.01$. The integration stops when the outer planet abandons the system after 8,103.2 years. The diagrams for GJ Set2-a of the evolution of the logarithm of the absolute error in total energy and in angular momentum of the system, the FLI and the semi major axes, the eccentricities and the AMD stability coefficients in Appendix B (Figures B.7, B.8, B.9, B.10 and B.11). In this case the system is also AMD - unstable. The absolute error in total energy is less than $10^{-14}$ and the absolute error in angular momentum is less than $10^{-15}$ before the abortion.

We also integrate the whole system from Table 3.2 with reference planet the
GJ 876 c (GJ Set2-b). The integration time step is $\tau = 0.01$. The integration stops when the outer planet abandons again the system after 14,269 years. All the diagrams referring to this system are in Appendix B (Figures B.12, B.13, B.14 and B.15). The absolute error in total energy is around $10^{-10}$ and the absolute error in angular momentum is less than $10^{-13}$ before the abortion.

The FLI is increasing rapidly in time in both GJ Set2-a and GJ Set2-b.

3.2 HR 8799

HR 8799 is located in the constellation of Pegasus, about 129 light years away from earth. The star is 1.5 times the Sun’s mass and 4.9 times its luminosity. This extrasolar system consists of a circumstellar disk comprised of two debris belts and four giant planets. The planets were discovered through direct imaging of the system. Three of them (HR 8799 b, c and d) were discovered in 2008 and the last and innermost planet (HR 8799 e) in 2010.

Gozdziewski et al. in [8] assume that the planets of the system are most likely trapped in a double Laplace resonance. The orbital elements of the system suggested from Gozdziewski et al. are presented at Table 3.3.

Table 3.3: Orbital Elements for HR 8799 (HR Set) For this set $M_*=1.56M_\odot$.
(Gozdziewski et al. [8])

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Planet e</th>
<th>Planet d</th>
<th>Planet c</th>
<th>Planet b</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass $(M_{Jup})$</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>$a$ (AU)</td>
<td>15.4</td>
<td>25.4</td>
<td>39.4</td>
<td>69.1</td>
</tr>
<tr>
<td>$e$</td>
<td>0.13</td>
<td>0.12</td>
<td>0.05</td>
<td>0.02</td>
</tr>
<tr>
<td>$i$ (°)</td>
<td>25</td>
<td>25</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>$\omega$ (°)</td>
<td>176</td>
<td>91</td>
<td>151</td>
<td>95</td>
</tr>
<tr>
<td>$\Omega$ (°)</td>
<td>64</td>
<td>64</td>
<td>64</td>
<td>64</td>
</tr>
<tr>
<td>$M$ (°)</td>
<td>326</td>
<td>58</td>
<td>148</td>
<td>321</td>
</tr>
</tbody>
</table>

By using the orbital elements from Table 3.3, we integrate the whole system (HR Set) for 10.1 million years with reference planet the HR 8799 e. The integration time step is $\tau = 0.01$. The absolute error in total energy and in angular momentum is less than $10^{-13}$ during the integration time (Figure 3.4). The FLI is almost stable as shown in Figure 3.5, which indicates a stable system. This is also confirmed from the diagram of AMD stability coefficients of the system (Figure 3.6). All coefficients are less than one and thus the system is characterized as
strong AMD stable despite the existence of resonance. The evolution of the semi major axes and the eccentricities are shown in Figures C.1 and C.2 respectively in Appendix C.

Figure 3.4: Diagrams of the logarithm of the absolute error in total energy vs time and the logarithm of the absolute error in angular momentum vs time for HR Set.

Figure 3.5: Diagrams of the FLI vs time for HR Set.
3.3 TRAPPIST-1

TRAPPIST-1 was discovered in 2015. It is an ultra-cool red dwarf star in the size of Jupiter approximately, located 39.5 light-years from the Sun in the constellation Aquarius. There are seven planets, discovered between 2015 and 2017, orbiting the star, with three of them considered to be within its habitable zone. Those are the planets e, f and g. Five of the planets (b, c, e, f and g) are similar in size to Earth, and two (d and h) are intermediate in size between Mars and Earth.

In Table 3.4 are the orbital elements for all seven planets. In this study we run six sets of initial conditions for this planetary system. The first set (TR Set1) includes the Planets b and c. The second set (TR Set2) includes the Planets b, c and d. The third set (TR Set3) includes the Planets b, c, d and e. The fourth set (TR Set4) includes the Planets b, c, d, e and f. The fifth set (TR Set5) includes the Planets b, c, d, e, f and g. Finally, the last set (TR Set) includes all of the planets.

In each set the reference planet is TRAPPIST-1 b, and the integration time step is \( \tau = 0.03 \) and the total integration time is \( 6.21 \cdot 10^5 \) years.

The diagrams for TR Set1 to TR Set5 are presented in Appendix D. In the next pages we are going to present the diagrams from the integration of the whole system (TR SET).

Figure 3.6: Diagram of the AMD stability coefficients \( \beta \) vs time for HR set.
Table 3.4: *Orbital Elements for TRAPPIST-1 (TR Set)*. For this set $M_*=0.08M_\odot$, $i=0^\circ$ instead of the inclinations presented in Exoplanets.eu [23]. We also set the mass of TRAPPIST-1 f equal to $0.003M_{Jup}$ and its eccentricity equal to zero.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Planet b</th>
<th>Planet c</th>
<th>Planet d</th>
<th>Planet e</th>
<th>Planet f</th>
<th>Planet g</th>
<th>Planet h</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass ($M_{Jup}$)</td>
<td>0.0027</td>
<td>0.00434</td>
<td>0.0013</td>
<td>0.002</td>
<td>0.0021</td>
<td>0.00422</td>
<td>0.003</td>
</tr>
<tr>
<td>a (AU)</td>
<td>0.01111</td>
<td>0.01521</td>
<td>0.02144</td>
<td>0.02817</td>
<td>0.0371</td>
<td>0.0451</td>
<td>0.063</td>
</tr>
<tr>
<td>$e$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$i$ (°)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$\omega$ (°)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$\Omega$ (°)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$M$ (°)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
3.3. TRAPPIST-1

In each set of initial conditions the errors in total energy of the system and in angular momentum never exceed $10^{-14}$ (Figures D.1, D.5, D.10, D.15, D.20, 3.7). The FLI for TR Set1 and TR Set2 is increasing in time but still remains small (Figures D.2, D.6). The last registered value of FLI for TR Set1 is 4.334334 and the same value for TR Set2 is 4.330330. For all the other sets of initial conditions the FLI is increasing rapidly in time (Figures D.11, D.16, D.21, 3.8).

The evolution of the semi major axes in time is stable in all cases (Figures D.3,
Figure 3.9: Diagram of semi major axes (a) vs time for TR Set

The values of AMD stability coefficients of the pairs of planets are always very small, less than one, in the sets of initial conditions we calculated them (Figures D.9, D.14, D.19, D.24, 3.11). Thus, TRAPPIST-1 is a strong AMD stable system. We also notice that the evolution of AMD stability coefficients presents the same behaviour as the evolution of eccentricities. That becomes obvious from Figures 3.10 and 3.11 from the TR Set of the seven planets.
Figure 3.10: Diagram of eccentricities (e) vs time for TR Set.
Figure 3.11: Diagram of the AMD stability coefficients $\beta$ vs time for TR Set.
Chapter 4

Conclusions

In this study we have completed a symplectic integration of a planetary system consists of N - planets. Along with the equations of motions, we solved the variational equations as well. The deviation vector is used in order to compute indicators to detect chaos such as Lyapunov Exponent and Fast Lyapunov Indicator.

The comparison between the symplectic integrator we used and the Bulirsch-Stoer method revealed that symplectic integrator is slower. However, the error in total energy for the symplectic integrator is significantly smaller in the majority of the cases we examined. The same results arise for the error in angular momentum of the system. In the last case where the errors are close, the Bulirsch-Stoer is not able to preserve it small. For the comparison of the two methods we used a stable system consists of four bodies, the star and three planets (Orbit 1, Table 2.1).

We developed a code based on a 6th order symplectic integrator in order to use it for the study of the long term evolution of the multipanet extrasolar systems. We executed several numerical integrations using different sets of initial conditions as presented in Chapter 3.

The existence of chaotic orbits does not necessarily lead to the destruction of the system. A chaotic system can be either AMD - unstable or AMD - stable as we obtain from the results of the systems GJ 876 and TRAPPIST-1 respectively. If a pair of planets is unstable does not indicate the collision between the two planets at least for the total time of integration we run. Note that a regular system can only be AMD - stable (HR 8799).

By using the tools developed in this study we can proceed to an extensive dynamical analysis of the extrasolar systems and monitor both FLI and AMD - stability coefficients for the different maps of initial conditions.

One more idea for future research is the comparison between AMD stability criterion and Hill stability criterion applied on the same systems.
Appendices
Appendix A

Diagrams for Comparison
Figure A.1: Diagrams of (a) the logarithm of the absolute error in total energy vs time and (b) the logarithm of the absolute error in angular momentum vs time for Orbit 1. (Case 2)
Figure A.2: Diagrams of (a) the logarithm of the absolute error in total energy vs time and (b) the logarithm of the absolute error in angular momentum vs time for Orbit 1. (Case 3)
Figure A.3: Diagrams of (a) the logarithm of the absolute error in total energy vs time and (b) the logarithm of the absolute error in angular momentum vs time for Orbit 1. (Case 4)
Appendix B

Diagrams for GJ 876

Figure B.1: Diagram of semi major axes (a) vs time for GJ Set1-a.
Figure B.2: Diagram of eccentricities ($e$) vs time for GJ Set1-a.

Figure B.3: Diagrams of the logarithm of the absolute error in total energy vs time and the logarithm of the absolute error in angular momentum vs time for GJ Set1-b.
Figure B.4: Diagram of the FLI vs time for GJ Set1-b.

Figure B.5: Diagram of semi major axes (a) vs time for GJ Set1-b.
Figure B.6: Diagram of eccentricities ($e$) vs time for GJ Set1-b.

Figure B.7: Diagrams of the logarithm of the absolute error in total energy vs time and the logarithm of the absolute error in angular momentum vs time for GJ Set2-a.
Figure B.8: Diagram of the FLI vs time for GJ Set2-a.

Figure B.9: Diagram of semi major axes (a) vs time for GJ Set2-a.
Figure B.10: Diagram of eccentricities (e) vs time for GJ Set2-a.

Figure B.11: Diagram of the AMD stability coefficients $\beta$ vs time for GJ Set2-a.
Figure B.12: Diagrams of the logarithm of the absolute error in total energy vs time and the logarithm of the absolute error in angular momentum vs time for GJ Set2-b.

Figure B.13: Diagram of the FLI vs time for GJ Set2-b.
Figure B.14: Diagram of semi major axes (a) vs time for GJ Set2-b.

Figure B.15: Diagram of eccentricities (e) vs time for GJ Set2-b.
Appendix C

Diagrams for HR 8799

Figure C.1: Diagram of semi major axes (a) vs time for HR Set.
Figure C.2: *Diagram of eccentricities (e) vs time for HR Set.*
Appendix D

Diagrams for TRAPPIST-1

Figure D.1: Diagrams of the logarithm of the absolute error in total energy vs time and the logarithm of the absolute error in angular momentum vs time for TR Set1.
Figure D.2: Diagram of the FLI vs time for TR Set1.

Figure D.3: Diagram of semi major axes (a) vs time for TR Set1
Figure D.4: *Diagram of eccentricities (e) vs time for TR Set1.*

Figure D.5: *Diagrams of the logarithm of the absolute error in total energy vs time and the logarithm of the absolute error in angular momentum vs time for TR Set2.*
Figure D.6: Diagram of the FLI vs time for TR Set2.

Figure D.7: Diagram of semi major axes (a) vs time for TR Set2.
Figure D.8: Diagram of eccentricities (e) vs time for TR Set2.

Figure D.9: Diagram of the AMD stability coefficients β vs time for TR Set2.
APPENDIX D. DIAGRAMS FOR TRAPPIST-1

Figure D.10: Diagrams of the logarithm of the absolute error in total energy vs time and the logarithm of the absolute error in angular momentum vs time for TR Set3.

Figure D.11: Diagram of the FLI vs time for TR Set3.
Figure D.12: Diagram of semi major axes (a) vs time for TR Set3.

Figure D.13: Diagram of eccentricities (e) vs time for TR Set3.
Figure D.14: Diagram of the AMD stability coefficients $\beta$ vs time for TR Set3.

Figure D.15: Diagrams of the logarithm of the absolute error in total energy vs time and the logarithm of the absolute error in angular momentum vs time for TR Set4.
Figure D.16: Diagram of the FLI vs time for TR Set4.

Figure D.17: Diagram of semi major axes (a) vs time for TR Set4
Figure D.18: Diagram of eccentricities (e) vs time for TR Set4.

Figure D.19: Diagram of the AMD stability coefficients β vs time for TR Set4.
Figure D.20: Diagrams of the logarithm of the absolute error in total energy vs time and the logarithm of the absolute error in angular momentum vs time for TR Set5.

Figure D.21: Diagram of the FLI vs time for TR Set5.
Figure D.22: Diagram of semi major axes (a) vs time for TR Set5

Figure D.23: Diagram of eccentricities (e) vs time for TR Set5.
Figure D.24: Diagram of the AMD stability coefficients $\beta$ vs time for TR Set5.
APPENDIX D. DIAGRAMS FOR TRAPPIST-1
Appendix E

Code in C++

For both methods used in this study, we read the same file of initial conditions. Some of the functions run in the program are the same and contain information for both of them.

Initial Conditions File

This is an example of the file of initial conditions we read for both of the methods. This example contains the initial conditions of Orbit 1, Table 2.1.

```
// Orbit 1 – (Voyatzis)
// configuration parameters + initial conditions

Number of Bodies (without SUN) = 3
Integration accuracy = 1.0e-12
Integration minstep = 0.0001
Integration maxstep = 0.1
Integration start_step = 0.01

Masses and Initial Conditions
Mass of Star M = 1.0

Period of 1st Planet in the list (Physical units) TP1=1
Basic time step (in evolutions TP1 of 1st planet) DT1=100
Number of DT1 iterations NN=40000
Screen output per L=100

Mass (Msun) a(AU) ecc incl M om OM=
0.001 1.00000 0.6950 0.00 0.00 0.0 0.0
0.002 1.58900 0.3410 0.00 0.00 0.0 0.0
0.0005 2.52800 0.1390 0.00 0.00 0.0 0.0
```
Initialization of the System

This function is called after reading the file of initial conditions. It takes as arguments the number of bodies, the vector of the masses and a BSVAR structure which contains information about the integration method. For the symplectic integrator it contains the integration time step $\tau$. For Bulirsch - Stoer the integration accuracy, the integration minimum time step, the integration maximum time step and the integration starting time step. The integration time step of the symplectic method is the integration starting time step of the Bulirsch - Stoer method.

With this function we initialize all the main and auxiliary vectors and matrices of our system of equations. Note that

- $NB$ is the number of planets
- $NDOF$ is the number of degrees of freedom of the system
- $NEQ$ is the number of differential equations of first order of the system

```cpp
int InitDSys(int Nbodies, double m[], BSVAR bs) {
    NB=Nbodies;
    NDOF=NB*3;
    NEQ=NDOF*4+1;
    mas=new double[NB+1]; // masses [mas[0]]=mass of S-BODY
    mas0 = new double[NB + 1]; // mas[0]+mas[i], 0: useless
    DfSi=new double[NDOF+1]; // Variable differences from S–Body (XS–Xi)
    //
    for(int k=0; k<3; k++) Dkij[k]=Alloc2Dspace(NB+1,NB+1);
    // Difs of k–var between i and j body e.g. D(1,2,4)=Y2–Y4
    //
    rSi3=new double[NB+1]; // distances of bodies from S–BODY ^3
    rBij3=Alloc2Dspace(NB+1,NB+1); // distances between bodies ^3
    rSi5=new double[NB+1]; // distances of bodies from S–BODY ^5
    rBij5=Alloc2Dspace(NB+1,NB+1); // distances between bodies ^5
    if (Dkij==NULL || DfSi==NULL || rBij3==NULL || rBij5==NULL || mas==NULL
        || rSi3==NULL || rSi5==NULL) return 0;
    mtot=0;
    // interaction configuration
    ss=Alloc2DspaceIntegers(NB+1,NB+1);
}
for (int i = 1; i <= NB; i++) for (int j = 1; j <= NB; j++) ss[i][j] = 1; // all
for (int i = 1; i <= NB; i++) ss[i][i] = 0; // except same bodies
for (int i = 0; i <= NB; i++) { mas[i] = m[i]; mas0[i] = m[0] + m[i]; mtot += m[i];}

if (CMFREE)
{
    for (int i = 1; i <= NDORF; i++) DfSi[i] = 0;
    for (int i = 1; i <= NB; i++) {rSi3[i] = 1; rSi5[i] = 1;} // unused
    for (int k = 0; k < 3; k++)
    for (int i = 0; i <= NB; i++)
    for (int j = 0; j <= NB; j++) Dkij[k][i][j] = 0;
    for (int i = 1; i <= NB; i++)
    {rBij3[i][i] = 1; rBij5[i][i] = 1;} // zero distances between the
    same bodies – unused nonzero values
}

pDerivij = Alloc2Dspace(NDOF+1,NDOF+1); // partial derivatives of
coeffs of variational equations
if (pDerivij == NULL) return 0;
derivs = dSystem2NBSV;
if (! InitSI(NEQ, bs)) return 0; // InitBs2(NEQ, bs)
return 1;
}

Calculation of $\frac{\partial V}{\partial \mathbf{q}}$ and $D^2_V(\mathbf{q}) \cdot \delta(\mathbf{q})$

This function is common in both methods. It is called any time we need to calculate the quantities $\frac{\partial V}{\partial \mathbf{q}}$ and $D^2_V(\mathbf{q}) \cdot \delta(\mathbf{q})$ to solve the equations of motion and the variational equations of the system. It takes as arguments the time, the vector $X[NEQ + 1]$ which contains all the Cartesian coordinates for the orbits $\mathbf{q}_i = \{x_i, y_i, z_i, p_{ix}, p_{iy}, p_{iz}\}$ and the deviation vector $\delta \mathbf{q}_i = \{\delta x_i, \delta y_i, \delta z_i, \delta p_{ix}, \delta p_{iy}, \delta p_{iz}\}$ with $0 \leq i \leq N$ as explained below:

- $X[0]$ is the time
- from $X[1]$ to $X[NDOF]$ the coordinates $x, y, z$ of the bodies are stored as $\{x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_N, y_N, z_N\}$
- from $X[NDOF + 1]$ to $X[2 \cdot NDOF]$ the coordinates $\delta x, \delta y, \delta z$ of the planets are stored as $\{\delta x_1, \delta y_1, \delta z_1, \delta x_2, \delta y_2, \delta z_2, \ldots, \delta x_N, \delta y_N, \delta z_N\}$
• from $X[2 \cdot NDOF + 1]$ to $X[3 \cdot NDOF]$ the coordinates $p_x, p_y, p_z$ of the planets are stored as \{\(p_{1x}, p_{1y}, p_{1z}, p_{2x}, p_{2y}, p_{2z}, \ldots, p_{Nx}, p_{Ny}, p_{Nz}\)\}

• from $X[3 \cdot NDOF + 1]$ to $X[4 \cdot NDOF]$ the coordinates $\delta p_x, \delta p_y, \delta p_z$ of the planets are stored as \{\(\delta p_{1x}, \delta p_{1y}, \delta p_{1z}, \delta p_{2x}, \delta p_{2y}, \delta p_{2z}, \ldots, \delta p_{Nx}, \delta p_{Ny}, \delta p_{Nz}\)\}

The elements of vector $dd[2 \cdot NDOF + 1]$ are calculated in this function.

• from $dd[1]$ to $dd[NDOF]$ contains the derivatives \(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial y_1}, \ldots, \frac{\partial V}{\partial x_N}, \frac{\partial V}{\partial y_N}, \frac{\partial V}{\partial z_N}\)\)

• from $dd[NDOF + 1]$ to $dd[2 \cdot NDOF]$ contains the coefficients of the variational equations for each body in row as presented in (I.31).

```cpp
void dSystem2NBSV(double t, double X[], double dd[])
{
    // S–BODY position;
    if (!CMFREE)
    {
        double XS[3]; // X,Y,Z
        for (int k=0; k<3; k++)
        {
            double sum=0;
            for (int i=1; i<=NB; i++) sum+=mas[i]*X[SPosIndex(i,k)];
            XS[k]=sum/mas[0];
        }
        for (int I=1; I<=NDOF; I++) DfSi[I]=XS[VarIndex(I)]-X[1]; // Var differencies with respect to S–BODY
    }
    // Var differencies among bodies
    for (int k=0; k<3; k++) for (int i=1; i<=NB; i++) for (int j=i+1; j<=NB; j++)
    {
        Dkij[k][i][j]=X[SPosIndex(i,k)]-X[SPosIndex(j,k)];
        Dkij[k][j][i]=-Dkij[k][i][j];
    }
    // functions of distances
    // A. from S–Body
    if (!CMFREE)
    {
        for (int i=0; i<NB; i++)
        {
            int k=3*i+1;
            double r2=DfSi[k]*DfSi[k]+DfSi[k+1]*DfSi[k+1]+DfSi[k+2]*DfSi[k+2]; // square
            rSi5[i+1]=1.0/(r2*r2*sqrt(r2)); // 1/r^5
            rSi3[i+1]=rSi5[i+1]*r2; // 1/r^3
        }
    }
    // B. between bodies
    for (int i=1; i<NB; i++)
    {
        for (int j=i+1; j<=NB; j++)
    }
```


```c
double r2 = Dkij[0][i][j] * Dkij[0][i][j] + Dkij[1][i][j] * Dkij[1][i][j] + Dkij[2][i][j] * Dkij[2][i][j];

double r5 = 1.0 / (r2 * r2 * sqrt(r2));

rBij5[i][j] = ss[i][j] * r5;
    rBij5[j][i] = ss[j][i] * r5;   // sij / r^5

rBij3[i][j] = rBij5[i][j] * r2;
    rBij3[j][i] = rBij5[j][i] * r2;   // sij * mi / r^3
```

// Differential canonical equations of 2ND ORDER [I=1 ... NDOF]
//

```c
for (int I = 0; I <= 2*NDOF; I++) dd[I] = 0; // Initialize derivs
if (!CMFREE)
{
    for (int I = 1; I <= NDOF; I++)
        dd[I] = mas[0] * rSi3[BodyIndex(I)] * DfSi[I];   // (mo/r^3)*(x0-xi)
}
```

// Coefficients of Variational equations NDOF x NDOF partialX [I]/partial X[J]
//

```c
// first part (M1) - S-BODY terms
for (int I = 1; I <= NDOF; I++)
    { int lb = BaseSposIndex(I);
      int i = BodyIndex(I);
      int vi = VarIndex(I);
      double fl = 3 * rSi5[i] * DfSi[l];
      for (int J = 1; J <= NDOF; J++)
          { int jb = BaseSposIndex(J);
            int j = BodyIndex(J);
            int vj = VarIndex(J);
```
\textbf{Initialization of the Symplectic Integrator}

This function is modified for each method. It is called in the function of the initialization of the symplectic integration. It takes as arguments the number of equations and the BSVAR structure which contains information about the integration
method. For the symplectic integrator we only care about the variable $Dt0$ of the BSVAR structure which is the integration time step.

In this function we initialize the two vectors containing the coefficients $c_i$ and $d_i$ (2.24). We also initialize two auxiliary vectors $TM1$ and $TM2$.

```c
int InitSI(int numeq, BSVAR bs)
{
    // Acc = bs.acc;
    // BSMinStep = bs.dtmin;
    // BSMaxStep = bs.dtmax;
    Dt0 = bs.dt0;
    neq = numeq;
    ndof = (neq - 1) / 4;
    ndof2 = 2 * ndof;
    double w1, w2, w3, w0;
    // constants 6th
    w1 = -1.177679984178877203859769906557630747557;
    w2 = 0.235532133593562427353873545143869705498;
    w3 = 0.784513610477561851297423345386050641537;
    w0 = (1. - 2. * (w1 + w2 + w3));
    d6[0] = w3; c6[0] = w3 / 2.;
    d6[1] = w2; c6[1] = (w3 + w2) / 2.;
    d6[2] = w1; c6[2] = (w2 + w1) / 2.;
    d6[3] = w0; c6[3] = (w1 + w0) / 2.;
    d6[4] = w1; c6[4] = c6[3];
    d6[5] = w2; c6[5] = c6[2];
    d6[6] = w3; c6[6] = c6[1];
    d6[7] = 0.; c6[7] = c6[0];
    TM1 = new double[neq];
    TM2 = new double[neq];
    SINITIALIZED = 1;
    return 1;
}
```

**Symplectic Integrator**

This is the function called from the main program when we want to integrate the system. It takes as arguments the vector of the $yin[N\text{EQ} + 1]$ with

- $yin[0]$ is the time
- from $yin[1]$ to $yin[N\text{DOF}]$ the coordinates $x, y, z$ of the bodies are stored as $\{x_1, y_1, z_1, x_2, y_2, z_2, ..., x_N, y_N, z_N\}$
from \( yin[NDOF + 1] \) to \( yin[2 \cdot NDOF] \) the coordinates \( \delta x, \delta y, \delta z \) of the planets are stored as \( \{ \delta x_1, \delta y_1, \delta z_1, \delta x_2, \delta y_2, \delta z_2, \ldots, \delta x_N, \delta y_N, \delta z_N \} \).

from \( yin[2 \cdot NDOF + 1] \) to \( yin[3 \cdot NDOF] \) the coordinates \( p_x, p_y, p_z \) of the planets are stored as \( \{ p_{1x}, p_{1y}, p_{1z}, p_{2x}, p_{2y}, p_{2z}, \ldots, p_{Nx}, p_{Ny}, p_{Nz} \} \).

from \( yin[3 \cdot NDOF + 1] \) to \( yin[4 \cdot NDOF] \) the coordinates \( \delta p_x, \delta p_y, \delta p_z \) of the planets are stored as \( \{ \delta p_{1x}, \delta p_{1y}, \delta p_{1z}, \delta p_{2x}, \delta p_{2y}, \delta p_{2z}, \ldots, \delta p_{Nx}, \delta p_{Ny}, \delta p_{Nz} \} \).

the time of integration \( T \) and the integration time step \( dt \).

The argument \( yout \) is the vector of all the variables of the system after the integration. The integration covers the time span from \( yin[0] \) to \( yin[0] + T \) with time step \( dt \).

```cpp
int Symplectic6thRUN(double yin[], double T, double dt, double yout[])
{
    double *y = TM1;
    for (int i = 0; i < neq; i++) y[i] = yin[i];
    double t0 = y[0];
    int n = (int)(T / dt);
    for (int i = 0; i < n; i++)
    {
        Symplectic6thSTEP(y, dt);
    }
    Symplectic6thSTEP(y, t0 + T - y[0]);
    for (int i = 0; i < neq; i++) yout[i] = y[i];
    return 1;
}
```

**Symplectic Integrator Step**

In this function we execute the intermediate steps for the positions \( q, \delta q \) and the velocities \( p, \delta p \) of the system. The structure of vector \( y \) is the same as the one of \( yin \) in previous function. The argument \( dt \) is the time step. The integration covers the time span from \( y[0] \) to \( yin[0] + dt \).

```cpp
void Symplectic6thSTEP(double y[], double dt)
{
    double *dxdy = TM2;
    for (int i = 0; i < 7; i++) // c[i], d[i]
    {
        symplectic_6_pos(y, dt, i); // i -> c[i]
        derivs(y[0], y, dxdy); // calculation of the derivatives coefficients for velocities
        symplectic_6_vel(y, dxdy, dt, i); // i -> d[i]
    }
}
```
symplectic_6_pos(y, dt, 7); // 7 => c[7]
y[0] += dt;
APPENDIX E. CODE IN C++
Bibliography


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