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Nonlinear electromagnetic metamaterials: aspects on mathematical modeling and physical phenomena

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Abstract

Electromagnetic metamaterials with simultaneously negative effective dielectric permittivity \( \varepsilon \) and magnetic permeability \( \mu \) (double-negative or left-handed or negative-index metamaterials) have a long history going back to the seminal paper of Veselago in 1968. Such metamaterials exhibit unusual and remarkable effects, like, e.g., the reversal of Snell’s law, the support of backward-propagating waves, and the possibility of obtaining a perfect lens. Nonlinear left-handed metamaterials are also very useful in tunable structures with intensity-controlled transmission and in switching the material properties from left- to right-handed and back. In this paper, we present aspects on mathematical modeling and physical phenomena governing electromagnetic wave propagation in nonlinear metamaterials. The metamaterials under consideration are described by a Drude-Lorentz frequency model of the linear parts and a Kerr-type behavior of the nonlinear parts of \( \varepsilon \) and \( \mu \), respectively. We show that in the left-handed band of the metamaterial wave propagation is governed by a higher-order nonlinear Schrödinger (NLS) equation, and derive analytically ultra-short bright or dark solitons solutions. Then, we investigate wave propagation in the frequency band gaps, i.e., in the frequency regimes where \( \varepsilon \) and \( \mu \) have different signs, and, hence, linear waves are evanescent. In these band gaps localization of waves is possible if a nonlinearity is induced in the metamaterial medium. We derive a dissipative short-pulse equation (DSPE) governing ultrashort pulses that may be formed in the band gaps and discuss its solitons solutions.

Key words. nonlinear electromagnetic metamaterials, left-handed media, negative refractive index, frequency band gaps, solitons.
1 Introduction

Exactly fifty years have passed from the original seminal paper of Veselago [1] on the electrodynamic behavior of media with simultaneously negative values of dielectric permittivity $\varepsilon$ and magnetic permeability $\mu$, where remarkable and extraordinary phenomena were predicted to occur in such double-negative (DNG) media. These phenomena include negative refraction, reversal of conventional Snell’s law, reversal of Doppler and Vavilov-Cerenkov effects, and backward waves propagation. For these reasons, Veselago referred to such materials as left-handed. Nevertheless, the important findings of [1] were not significantly exploited until composite DNG media were experimentally fabricated by Smith et al in 2000 [2]. Then, the fascinating research field of metamaterials was gradually established, the theoretical and experimental interest revived, and new important and unconventional physical phenomena were reported. Pendry suggested that a negative refractive index medium can operate as a perfect lens focusing propagating and evanescent electromagnetic waves [3]. In 2001, Shelby, Smith, and Schultz demonstrated experimentally the negative index of refraction and the reversal of Snell’s law in a composite metamaterial medium [4]. Ziolkowski and Heyman demonstrated that a DNG slab can act as a convertor from a pulsed cylindrical or spherical wave to a pulsed beam [5]. Engheta proposed to exploit the optical properties of metamaterials composed of nanoparticles playing the role of lumped nanocircuit elements (analogous to microelectronics) in order to obtain circuits for optical information processing at the nanometer scale [6]. Collections of the important derived physical results and the presented experimental configurations from these early years of metamaterials can be found in the books [7]-[9].

Metamaterials are most commonly defined as artificially constructed materials having electromagnetic properties not generally found in nature; see e.g. [10]. Despite the fact that metamaterials were initially identified with DNG (negative-refraction) media, over the years the associated research field has broadened so much as to include now any periodic or even non-periodic configurations of natural materials, which overall produce desired physical properties not found in nature. Usually, the scales of the constituent materials are smaller than the wavelengths of interest. Additionally, metamaterials are no more limited to application domains related exclusively to electromagnetic waves, but cover also research areas associated to acoustic,
elastic waves propagation and so on. The research interest in metamaterials still exhibits exponential growth and it is indicative that there are scientific journals as well as conference series that are devoted entirely to relevant topics.

The properties of the DNG materials were initially studied in the linear regime of electromagnetic wave propagation, where both $\varepsilon$ and $\mu$ are assumed to be independent of the intensity of the electromagnetic fields. However, modeling tunable structures where the field intensity changes the transmission properties of the composite structure would require the investigation of nonlinear properties of such metamaterials. To this end, Zharov, Shadrivov, and Kivshar analyzed, in 2003, nonlinear properties of left-handed metamaterials composed of a lattice of split-ring resonators and wires embedded in a nonlinear dielectric \[11\]. They predicted the dependence of the effective magnetic permeability on the intensity of the magnetic field which subsequently enables the switching between the left- and right-handed materials by varying the field intensity. Experimental realization of a fully controlled, tunable nonlinear metamaterial system was presented in \[12\]. Precisely, variable-capacitance diodes were added to split-ring resonator (SRR) structures and changes in the diode capacitance adjusted the effective magnetic permeability of left-handed metamaterials and triggered self-induced nonlinear effects. These findings stimulated further theoretical and experimental work in the field of tunable nonlinear metamaterials; see e.g. the discussion in \[13\]. Nonlinear electromagnetic metamaterial structures have been shown to be particularly useful in several applications, including, e.g., reconfigurable refractive index systems \[14\], creating controllable shielding effects, accompanied by a parametric reflection \[15\], focusing of second-harmonic signals with nonlinear metamaterial lenses \[16\], and propagation of temporal rogues and solitons in hyperbolic metamaterials waveguides \[17\].

In this paper, we consider metamaterials described by a Drude-Lorentz frequency model of their effective permittivity and permeability, respectively. Such metamaterials possess four frequency bands: a right-handed band (where the permittivity and permeability are both positive), a left-handed band, as well as two electromagnetic band gaps (EBGs), where linear waves are evanescent. We overview some of the fundamental physical phenomena occurring in (linear) left-handed metamaterials as well as the relevant experimental realizations. Then, we focus on nonlinear electromagnetic metamaterials and first present the initial theoretical and experimental works for the investigations and realizations of tunable left-handed metamaterials. Main emphasis is given on the mathematical methodologies applied for the analysis of the electromagnetic wave
propagation phenomena in nonlinear left-handed metamaterials. We show that wave propagation is governed by a higher-order nonlinear Schrödinger (NLS) equation, describing ultra-short solitons in nonlinear left-handed metamaterials. In fact, the derived NLS equations generalize the ones describing short pulse propagation in nonlinear optical fibers. Analyzing these equations, we find necessary conditions for the formation of bright or dark solitons and derive analytically ultra-short solitons solutions. In the sequel, we investigate wave propagation in the EBGs, i.e., in the frequency regimes where \( \varepsilon \) and \( \mu \) have different signs. In these band gaps, localization of waves is possible if a nonlinearity is induced in the metamaterial. We start from Maxwell’s equations and derive nonlinear evolution equations governing ultrashort gap pulses, which may be formed in the EBGs. Precisely, we use a multi-scale expansion method and derive a dissipative short-pulse equation (DSPE), that describes few-cycle pulses and also incorporates linear loss (the loss terms are found in terms of the metamaterials parameters). Different gap solitons solutions of the short-pulse equations are discussed. The effect of dissipation in the propagation of gap solitons is briefly analyzed and techniques for compensating losses are suggested.

2 Fundamental Physical Phenomena in Left-Handed Metamaterials and Experimental Realizations

The frequency-dependent functions of the (linear) effective dielectric permittivity and magnetic permeability can be characterized by different physical models. Here, we consider the so-called Drude-Lorentz model (see, e.g., [18]-[20]) in the framework of which we have

\[
\hat{\varepsilon}_L(\omega) = \varepsilon_0 \left( 1 - \frac{\omega_p^2}{\omega^2 + i\omega\gamma_\varepsilon} \right),
\]

\[
\hat{\mu}_L(\omega) = \mu_0 \left( 1 - \frac{F\omega^2}{\omega^2 + i\omega\gamma_\mu - \omega_{res}^2} \right),
\]

where \( \varepsilon_0 \) and \( \mu_0 \) are the vacuum permittivity and permeability, \( \omega_p \) is the plasma frequency, \( F \) is the filling factor, \( \omega_{res} \) is the linear resonant split-ring resonator (SRR) frequency, while \( \gamma_\varepsilon \) and \( \gamma_\mu \) are the linear loss frequencies. Typical dispersion curves of \( \text{Re}[\hat{\varepsilon}_L(\omega)] \) and \( \text{Re}[\hat{\mu}_L(\omega)] \), obeying the Drude-Lorentz model of Eqs. (2.1) and (2.2), are shown in Fig. 1a, while the respective imaginary parts \( \text{Im}[\hat{\varepsilon}_L(\omega)] \) and \( \text{Im}[\hat{\mu}_L(\omega)] \) are depicted in Fig. 1b (see the related
When $\omega_r > \omega_{\text{res}}$ and the linear losses $\gamma_\varepsilon$ and $\gamma_\mu$ are small compared to the operating frequency $\omega$, then linear waves propagate in two frequency bands in which $k_0^2 = \omega_0^2 \text{Re}[\hat{\varepsilon}_L(\omega_0)] \text{Re}[\hat{\mu}_L(\omega_0)] > 0$, where $k_0$ is the wavenumber corresponding to the carrier frequency $\omega_0$. Precisely, in the band $\omega > \omega_r$ holds $\text{Re}[\hat{\varepsilon}_L(\omega)] > 0$ and $\text{Re}[\hat{\mu}_L(\omega)] > 0$, and hence the effective medium is right-handed (RH), while in the band $\omega_{\text{res}} < \omega < \omega_m = \omega_{\text{res}} / \sqrt{1-F}$ holds $\text{Re}[\hat{\varepsilon}_L(\omega)] < 0$ and $\text{Re}[\hat{\mu}_L(\omega)] < 0$, and the medium is left-handed (LH). These bands will be called hereafter RH and LH bands. In both the RH and LH bands holds $\text{Re}[\hat{\varepsilon}_L(\omega)] \text{Re}[\hat{\mu}_L(\omega)] > 0$. The essential difference is that in the RH band the effective refractive index is positive, while in the LH band the effective refractive index is negative. For this reason, LH media are also called as negative-index media.

Besides, there exist two frequency regions, with $\text{Re}[\hat{\varepsilon}_L(\omega)] < 0$ and $\text{Re}[\hat{\mu}_L(\omega)] > 0$, where linear waves are evanescent. This is because $k_0^2 < 0$, and, thus, the wavenumber $k_0$ becomes purely imaginary. These frequency regions are called electromagnetic band-gaps (EBGs) and are defined by $0 < \omega < \omega_{\text{res}}$ [“low-frequency” (LF) gap], and $\omega_m < \omega < \omega_r$ [“high-frequency” (HF) gap]. Electromagnetic wave propagation and associated physical phenomena in the EBGs are analyzed in Sec. 4 below.

Figure 1: Real (a) and imaginary (b) parts of the relative linear magnetic permeability, $\hat{\mu}_L / \mu_0$ [solid (red) lines], and dielectric permittivity, $\hat{\varepsilon}_L / \varepsilon_0$ [dashed (blue) lines] versus frequency $\omega$ in arbitrary units (a.u.) for the considered Drude-Lorentz metamaterial model [c.f. Eqs. (2.1) and (2.2)].

Double negative (DNG) materials, i.e. materials with simultaneously negative dielectric permittivity $\varepsilon$ and magnetic permeability $\mu$, were investigated theoretically for the first time by Veselago in his historic 1968 paper [1]. There, he concluded that in such negative refractive index materials electromagnetic waves can propagate and are governed by a well defined dispersion relation $k_0^2 = \omega_0^2 c \mu$. However, certain unusual and remarkable phenomena were predicted to
occur in these materials. One of the most extraordinary phenomena is that the energy flow, as dictated by the Poynting vector, would be in the opposite direction to the wave vector, and hence rays would travel in the opposite direction to waves (see Fig. 2a). Since, changing the sign of both \( \varepsilon \) and \( \mu \) is equivalent in Maxwell’s equations to changing the sign of the magnetic field but keeping the same wave vector, Veselago referred to such materials as left handed. An other remarkable phenomenon (also pointed out in [1]) is that at an interface between double positive (i.e. \( \varepsilon > 0 \) and \( \mu > 0 \)) and DNG materials light would refract in the unconventional direction relative to the normal, namely towards the same region of the plane that the incident wave, traveling in the double positive material, impinges on the interface (see Fig. 2b).

Figure 2: (a) In left-handed (negative-index) metamaterials, the energy flow is in the opposite direction to the wave vector (adapted from [20]) (b) Reversal of Snell’s law in left-handed media: refraction in a negative-index metamaterial versus that in a conventional (right-handed) material having a positive refractive index of the same absolute value.

In negative-index media, the group velocity is negative, i.e. \( v_g = \frac{\partial \omega}{\partial k} < 0 \), and hence dispersion is a prerequisite property of a medium for negative refraction [20],[22]. The above described remarkable phenomena of DNG media generated considerable interest at the time of [1], but the inability of realization due to the absence of natural materials with the DNG property led to an eventual neglect of the subject for nearly thirty years. Then, near the end of the 1990s, it was proposed that new electromagnetic properties can be realized by micro-structuring a material on a subwavelength scale [18],[23] (see also the review [19]). In 1999, the first experimental realization was reported by Smith et al. [2], who demonstrated a composite medium, based on a periodic array of interspaced conducting nonmagnetic split ring resonators and continuous wires, that exhibits a frequency region in the microwave regime with simultaneously negative values of effective \( \varepsilon \) and \( \mu \) (see Fig. 3a). Then, in 2001, the negative index of refraction of the composite medium as well as the predicted negative refraction angle of the composite medium were also verified experimentally [4] (see Fig. 3b).

These experimental realizations of negative-index metamaterials revived the interest in the field. New important theoretical and experimental results were established. Pendry proposed the
possibility that a DNG medium with a negative index of refraction can achieve a *perfect lens*, which focuses the entire spectrum of electromagnetic waves, both the propagating as well as the evanescent ones [3]. Ziolkowski and Heyman demonstrated that waves in general DNG media focus in a paraxial sense along the propagation axis, i.e., they coalesce into localized beam fields that channel the power flow [5]. In this way, a DNG slab can act like a convertor from a pulsed cylindrical or spherical wave to a pulsed beam. The design, fabrication and testing of DNG metamaterials in the microwave region were presented by Ziolkowski in [24]. Grbic and Eleftheriades reported experimental results demonstrating the ability of a planar left-handed microwave lens, with a relative refractive index of $-1$, to form images that overcome the diffraction limit [25]. The fabricated left-handed lens was a planar slab consisting of a grid of printed metallic strips over a ground plane, loaded with series capacitors (C) and shunt inductors (L) (see Fig. 3c). The novel physical phenomena and the related engineering explorations during these first years of the revival of the interest in DNG metamaterials were overviewed in [7]-[9].

Figure 3: (a) A single copper split ring resonator (SRR) utilized as the unit cell of the periodic composite medium with simultaneously negative permittivity and permeability (adapted from [2]) (b) The left-handed metamaterial sample used for the experimental demonstration of negative-index behavior. The sample consists of square copper SRRs and copper wire strips on fiber glass circuit board material (adapted from [4]). (c) The left-handed planar transmission-line lens used for forming images that overcome the diffraction limit (adapted from [25]).

Nowadays, the research field of metamaterials is very rich and has exhibited rapid advance both from a theoretical as well as from an experimental point of view. Many scientific journals are devoted entirely to the topic of metamaterials and additionally dedicated conference series (like e.g. the *Metamaterials* conference [26]) have been established. ¹ Most importantly, at the present time, the field of metamaterials has become so broad as to include any combinations of natural materials (in periodic arrangements or not) which produce desired physical properties not found in nature. Moreover, the concepts of metamaterials have been extended to acoustic, thermal, elastic,

¹ In fact, in the 2018 edition of the Metamaterials conference there has been a celebration for the 50 years anniversary of the seminal paper of Veselago [1] followed by a historical presentation of the evolution of the research field spanning from the early years of metamaterials to state-of-the-art applications.
and mechanical metamaterials.

3 Nonlinear Left-Handed Electromagnetic Metamaterials

3.1 Tunable Metamaterials and Experimental Realizations

Nonlinear properties of left-handed metamaterials were analyzed theoretically for the first time in [11] for a lattice of SRRs and wires with a nonlinear dielectric (see Fig. 4a). It was shown that the effective magnetic permeability depends on the intensity of the magnetic field allowing switching between the left- and right-handed materials by varying the fields intensities. Particularly, it was elaborated that the nonlinear response of such a composite structure stems by two different contributions. The first one is the electric field’s intensity-dependent part of the effective dielectric permittivity of the infilling dielectric. The second contribution comes from the lattice of resonators, since the SRR capacitance depends on the strength of the local electric field in a narrow slot. The intensity of the local electric field in the SRR gap depends on the electromotive force in the resonator loop, which is induced by the magnetic field. Therefore, the effective magnetic permeability should depend on the induced magnetic field.

Figure 4: (a) Schematic of the composite nonlinear metamaterial structure. (adapted from [11]) (b) Fabricated SRR and biasing circuitry used for direct voltage tuning of the resonant frequency in a varactor-loaded SRR resonator system (adapted from [12]). (c) Structure of nonlinear tunable metamaterial containing SRRs with variable-capacity diode (adapted from [13]). (d) Schematic of a varactor-loaded SRR system in parallel connection with an inductive coil (adapted from [27]).

One of the first steps toward the fabrication of a fully-controlled, tunable nonlinear metamaterial systems was made in [12] by exploiting the tunability and self-induced nonlinear response of a single SRR (see also [13]). A variable-capacitance diode (varicap) was added to the SRR resonant structure (see Figs. 4b and 4c). Tunability of the diode capacitance was achieved by varying the width of its specifically doped P-N junction, and associated depletion layer, through the application of a DC bias voltage, giving rise to huge self-induced nonlinear effects. Changes in the diode capacitance alter the resonant conditions of the SRR producing frequency shifts, which in turn adjust the effective magnetic permeability of left-handed metamaterials. Under this setting,
it was demonstrated experimentally that the eigenfrequencies of the resonators can be tuned over a wide frequency range, and the self-induced nonlinear effects (observed in the varactor-loaded SRRs structures) can appear at relatively low power levels.

Moreover, experimental realizations at microwave frequencies of the dynamic tunability of the magnetic resonance of a single nonlinear SRR with varactor diode with and without an inductive coil in parallel connection were reported in [27] (see Fig. 4d). It was shown that the coil changes the sign of the nonlinearity and eliminates the memory effect caused by charge accumulation across the varactor.

In [11] and [28]-[33], it has been established that nonlinear electromagnetic metamaterials can be characterized by frequency-dependent effective dielectric permittivity and magnetic permeability of the following form

\[
\hat{\varepsilon}(\omega | | E|^2) = \hat{\varepsilon}_L(\omega) + \varepsilon_{NL}(|E|^2),
\]

(3.1)

\[
\hat{\mu}(\omega | | H|^2) = \hat{\mu}_L(\omega) + \mu_{NL}(|H|^2),
\]

(3.2)

where \(E\) and \(H\) are the electric and magnetic field intensities, respectively. The nonlinear parts exhibit a Kerr-type behavior and are given by

\[
\varepsilon_{NL}(|E|^2) = \varepsilon_0\hat{\alpha} |E|^2,
\]

(3.3)

\[
\mu_{NL}(|H|^2) = \mu_0\hat{\beta} |H|^2,
\]

(3.4)

where \(\hat{\alpha}\) and \(\hat{\beta}\) are the Kerr coefficients for the electric and magnetic fields, respectively. Focusing and defocusing dielectrics correspond to \(\hat{\alpha} > 0\) and \(\hat{\alpha} < 0\) respectively. The nonlinear coefficient \(\hat{\beta}\) can be found via the dependence of \(\hat{\mu}\) on the magnetic field intensity [11].

### 3.2 Mathematical Modeling of Wave Propagation in Nonlinear Left-Handed Metamaterials

In this section, we will present some aspects of the mathematical analysis concerning the localization and propagation of electromagnetic waves in nonlinear left-handed metamaterials. To this end, we consider a \(x\)- (\(y\)-) polarized electric (magnetic) field travelling along the \(+\hat{z}\) direction in the LH band of the Drude-Lorentz metamaterial medium (namely for \(\omega_{res} < \omega < \omega_M\), when \(\text{Re}[\hat{\varepsilon}_L(\omega)] < 0\) and \(\text{Re}[\hat{\mu}_L(\omega)] < 0\); see Fig. 1a)

\[
E(z,t) = \hat{x}E(z,t), \quad H(z,t) = \hat{y}H(z,t),
\]

(3.5)
where
\[ [E(z,t), H(z,t)]^T = [q(z,t), p(z,t)]^T e^{i(k_0 z - \omega t)}, \]  
(3.6)

with \( q \) and \( p \) being the electric and magnetic fields envelopes, respectively.

Then, Faraday’s and Ampère’s laws in the time domain take the form
\[ \partial_z E = -\partial_t (\mu \ast H), \quad \partial_z H = -\partial_t (\varepsilon \ast E), \]  
(3.7)
where \(*\) denotes the convolution integral, i.e., \( f(t) \ast g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau \). Eqs. (3.7) may be used in either the RH or the LH regime of a metamaterial. Precisely, once the dispersion relation \( k_0 = k_0(a_0) \) and the evolution equations for the fields \( E \) and \( H \) are found, then \( k_0 < 0 \) \( (k_0 > 0) \) corresponds to the LH (RH) regime.

Evolution equations for the electromagnetic fields envelopes \( q \) and \( p \) are obtained by employing the reductive perturbation method [34, 35]. First, the slow variables are introduced
\[ Z = \varepsilon^2 z, \quad T = \varepsilon (t - k'_0 z), \]  
(3.8)
where \( k'_0 \equiv v_s^{-1} \) is the inverse of the group velocity (primes denote derivatives with respect to \( \omega_0 \)), while \( \varepsilon \) is a formal small parameter characterizing the temporal spectral width of the nonlinear term with respect to the spectral width of the quasi-plane-wave dispersion relation [36, 37]. Next, \( q \) and \( p \) are expressed as asymptotic expansions in terms of \( \varepsilon \),
\[ q(Z,T) = q_0(Z,T) + \varepsilon q_1(Z,T) + \varepsilon^2 q_2(Z,T) + \mathcal{O}(\varepsilon^3), \]  
(3.9)
\[ p(Z,T) = p_0(Z,T) + \varepsilon p_1(Z,T) + \varepsilon^2 p_2(Z,T) + \mathcal{O}(\varepsilon^3). \]  
(3.10)

The linear components \( \hat{E}_L \) and \( \hat{\mu}_L \) are decomposed into real and imaginary parts, as
\[ \hat{E}_L(\omega) = \hat{E}_R(\omega) - i\hat{E}_I(\omega), \quad \hat{\mu}_L(\omega) = \hat{\mu}_R(\omega) - i\hat{\mu}_I(\omega), \]  
(3.11)
where the imaginary parts are assumed to be \( \mathcal{O}(\varepsilon^2) \). The Kerr coefficients \( \hat{a} \) and \( \hat{b} \) are also assumed of order \( \mathcal{O}(\varepsilon^2) \) (in accordance with [28], [36], [38], and [39])
\[ \varepsilon_{NL}(|E|^2) = \varepsilon_0 \alpha \varepsilon^2 |q|^2, \quad \mu_{NL}(|H|^2) = \mu_0 \beta \varepsilon^2 |p|^2. \]  
(3.12)

By the above considerations, group velocity dispersion, nonlinearity and linear loss, are all assumed to be at the same order, \( \mathcal{O}(\varepsilon^2) \). As a result, linear loss will appear in the nonlinear evolution equations at \( \mathcal{O}(\varepsilon^2) \) and at \( \mathcal{O}(\varepsilon^3) \).

Combining (3.7) and (3.12) and using the analytical techniques described in detail in [21]
and [40] lead to a hierarchy of equations at various orders of \( \varepsilon \). Particularly, the leading-order equation at \( \mathcal{O}(\varepsilon^0) \) provides the linear dispersion relation

\[ k_0^2 = \alpha_0^2 \hat{\varepsilon}_R \hat{\mu}_R, \]  

(3.13)

with all functions of frequency evaluated at \( \omega_0 \). At \( \mathcal{O}(\varepsilon^1) \), one obtains the group velocity \( v_g = 1/k' \). Next, at \( \mathcal{O}(\varepsilon^2) \), the following nonlinear Schrödinger (NLS) equation is obtained

\[ i \partial_z \phi - \frac{1}{2} k_0' \hat{\varepsilon}_R^2 \phi + g |\phi|^2 \phi = -i \hat{\Gamma} \phi, \]

(3.14)

where \( \phi = q_0, k_0'' \) is the group-velocity dispersion coefficient, while \( \hat{\Gamma} \) and \( g \) are the linear loss and nonlinear coefficients

\[ \hat{\Gamma} = \frac{|k_0|}{2 \hat{\varepsilon}_R \hat{\mu}_R} (\hat{\varepsilon}_R \hat{\mu}_I + \hat{\mu}_R \hat{\varepsilon}_I) \quad \text{and} \quad g = \frac{\alpha_0^2}{2k_0} \varepsilon_0 \alpha \hat{\mu}_R + \mu_0 \beta \hat{\varepsilon}_R \hat{Z}_L^{-2}), \]

(3.15)

where \( \hat{Z}_L = \sqrt{\hat{\mu}_R / \hat{\varepsilon}_R} \) is the linear wave-impedance.

Finally, at \( \mathcal{O}(\varepsilon^3) \), a higher-order NLS (HNLS) equation is obtained, incorporating higher-order dispersive and nonlinear terms. This equation involves the new combined function \( \Phi = \phi + \varepsilon \psi \) and is given by

\[ i \partial_z \Phi - \frac{1}{2} k_0' \hat{\varepsilon}_R^2 \Phi + g |\Phi|^2 \Phi = i \varepsilon \left[ \frac{1}{6} k_0'' \hat{\varepsilon}_R^3 \Phi - g' \hat{\varepsilon}_R (|\Phi|^2 \Phi) - \delta \Phi \hat{\varepsilon}_R (|\Phi|^2) \right] - i \hat{\Gamma} \Phi, \]

(3.16)

where

\[ \delta = \omega_0 \hat{Z}_L' \varepsilon_0 (\alpha - \mu_0 \beta \hat{Z}_L^{-4}). \]

(3.17)

For \( \varepsilon = 0 \), the HNLS Eq. (3.16) is reduced to the NLS Eq. (3.14), while for \( \varepsilon \neq 0 \) Eq. (3.16) generalizes the HNLS equation describing ultra-short pulse propagation in optical fibers [36],[37].

Eqs. (3.14) and (3.16) are reduced to dimensionless forms as follows: we measure length, time, and the field intensity \( |\phi|^2 \) (respectively \( |\Phi|^2 \)) in units of the dispersion length \( L_d = t_0^2 / |k_0'| \), initial pulse width \( t_0 \), and \( L_d / |g| \), respectively, and obtain

\[ i \partial_z \phi - \frac{s}{2} \hat{\varepsilon}_R^2 \phi + \sigma |\phi|^2 \phi = -i \Gamma \phi, \]

(3.18)

and

\[ i \partial_z \Phi - \frac{s}{2} \hat{\varepsilon}_R^2 \Phi + \sigma |\Phi|^2 \Phi = i \left[ \delta_1 \hat{\varepsilon}_R^2 \Phi - \delta_2 \hat{\varepsilon}_R (|\Phi|^2 \Phi) - \delta_3 \Phi \hat{\varepsilon}_R (|\Phi|^2) \right] - i \Gamma \Phi, \]

(3.19)
where \( s = \text{sign}(k_0^s) \), \( \sigma = \text{sign}(g) \), and \( \Gamma = L_\beta \tilde{\Gamma} \), while the coefficients \( \delta_1 \), \( \delta_2 \), and \( \delta_3 \) are given by

\[
\delta_1 = \frac{\varepsilon}{6 t_0} \left| k_0^s \right|, \quad \delta_2 = \frac{g'}{|g| t_0}, \quad \delta_3 = \frac{\delta}{|g| t_0}.
\]  

(3.20)

It is worth mentioning that the lossless HNLS model was first derived in [36] as an appropriate model describing propagation of sub-picosecond pulses in optical fibers with a Kerr nonlinearity (see also [37]-[39] and references therein). It is also interesting to note that a similar HNLS equation can be used to model deep water waves at finite depth (see, e.g., [41] and references therein).

### 3.3 Solitons in Nonlinear Left-Handed Metamaterials

The NLS Eq. (3.18) admits bright (dark) soliton solutions for \( s\sigma = -1 \) \( (s\sigma = +1) \). Specific conditions for the formation of bright or dark solitons (BS or DS) for the NLS Eq. (3.18) for typical realizable values of the model’s parameters are analyzed in [40]. More precisely, as is depicted in Fig. 5, for \( F = 0.4 \) and \( \omega_p = 2\pi \times 10^6 \) GHz, holds \( s = +1 \) (i.e., \( k_0^s > 0 \)) when \( 2\pi \times 1.76 < \omega < 2\pi \times 1.87 \) GHz, while \( s = -1 \) (i.e., \( k_0^s < 0 \)) when \( 2\pi \times 1.45 < \omega < 2\pi \times 1.76 \) GHz in the LH regime. As concerns the parameter \( \sigma \), it can take either the value \( \sigma = +1 \) or \( \sigma = -1 \), depending on the magnitudes and signs of the Kerr coefficients \( \alpha \) and \( \beta \). The final conclusions regarding the conditions for the formation of bright or dark solitons are reported in Table 1.

Table 1: Conditions for the formation of bright or dark solitons (BS or DS) for the NLS Eq. (3.18) \[ Z_0 = \sqrt{\mu_0 / \varepsilon_0} \] is the vacuum wave-impedance.

| \( \sigma = +1 \) | \( \alpha > 0 \) | \( s = +1 \) | BS | \( s = -1 \) | DS |
| \( \sigma = -1 \) | \( \alpha < 0 \), \( \frac{|a|}{|b|} > \frac{z_i}{z_i} \) | BS | DS |
| \( \sigma = +1 \) | \( \alpha < 0 \), \( \frac{|a|}{|b|} < \frac{z_i}{z_i} \) | DS | BS |
For $\sigma = +1$, bright (dark) solitons occur in the anomalous (normal) dispersion regimes, i.e., for $k_0^* < 0$ ($k_0^* > 0$), respectively. On the other hand, $\sigma = -1$ for a defocusing dielectric $\alpha < 0$, with $|\alpha / \beta| > Z_0^2 / Z_L^4$ and, thus, bright (dark) solitons occur in the normal (anomalous) dispersion regimes. Importantly, note that the “flexibility” arising from the extra “degree of freedom” provided by the presence of dispersion and nonlinearity properties in the magnetic response of the left-handed metamaterial (missing in fiber optics), allows for the formation of bright (dark) solitons in the anomalous (normal) dispersion regimes for defocusing dielectrics.

Figure 5: (a) The linear parts of the relative magnetic permeability, $\hat{\mu}_L / \mu_0$ [solid (red) line], and the dielectric permittivity, $\hat{\varepsilon}_L / \varepsilon_0$ [dashed (blue) line] as functions of the (angular) frequency $\omega = 2\pi f$ with the ordinary (linear) frequency $f$ measured in GHz, for $F = 0.4$ and $\omega_p = 2\pi \times (10 \text{ GHz})$. In the band $\omega_{\text{res}} = 2\pi \times (1.45 \text{ GHz})$ to $\omega_m = 2\pi \times (1.87 \text{ GHz})$ both $\hat{\mu}_L$ and $\hat{\varepsilon}_L$ are negative and, thus, the medium is left-handed. (b) The group-velocity dispersion (GVD) coefficient $k'' \equiv \partial^2 k / \partial \omega^2$ as a function of the frequency $\omega = 2\pi f$ with $f$ measured in GHz in the left-handed regime (adapted from [40]).

Next, we consider the HNLS Eq. (3.19), which may be used to model ultra-short soliton pulses in the LH frequency band of a nonlinear DNG metamaterial, in the framework of the slowly-varying envelope approximation. These pulse-shaped solutions feature widths of the extent of many (of the order of 10 or even 100) carrier periods [21].

More precisely, in the absence of losses, we seek travelling-wave solutions of Eq. (3.19) of the form,

$$ \Phi(Z,T) = U(\eta) \exp[i(KZ - \Omega T)], $$

(3.21)

where $U(\eta)$ is the unknown real envelope function, $\eta = T - AZ$, and the real parameters $A, K$ and $\Omega$ denote, respectively, the inverse velocity, wavenumber and frequency of the travelling wave. Substituting Eq. (3.21) into Eq. (3.19) and imposing certain compatibility conditions in the real and imaginary parts of the resulting equation, we obtain the following equation of motion of the unforced and undamped Duffing oscillator,

$$ \ddot{U} + \kappa U + \nu U^3 = 0, $$

(3.22)

where
\[
\kappa \equiv \frac{K - \frac{i}{2} \Omega^2 - \delta_1 \Omega^3}{\frac{i}{2} + 3 \delta_1 \Omega} = \frac{\Lambda - s \Omega - 3 \delta_1 \Omega^2}{\delta_1} \, ,
\]
(3.23)

\[
\nu = -\frac{\sigma \delta_2}{\delta_1} = -\frac{\sigma(1 + \delta_2 \Omega)}{\frac{i}{2} + 3 \delta_1 \Omega} \, .
\]
(3.24)

For \( \kappa \nu < 0 \), Eq. (3.22) possesses two exponentially localized solutions, corresponding to the separatrices in the \((U, \dot{U})\) phase-plane. These solutions have the form of a hyperbolic secant (tangent) for \( \kappa < 0 \) and \( \nu > 0 \) \((\kappa > 0 \) and \( \nu < 0 \)), thus corresponding to the bright, \( U_{bs} \) (dark, \( U_{ds} \)) solitons of Eq. (3.19), and given by

\[
U_{bs}(\eta) = (2 |\kappa|/|\nu|)^{1/2} \text{sech}(\sqrt{|\kappa||\eta|}),
\]
(3.25)

\[
U_{ds}(\eta) = (2|\kappa|/|\nu|)^{1/2} \text{tanh}(\sqrt{|\kappa|/2|\eta|}),
\]
(3.26)

Importantly, these are \textit{ultra-short solitons}\ of the HNLS Eq. (3.19). For \( \omega_0 t_0 = \mathcal{O}(1) \), or for soliton widths \( t_0 \sim \omega_0^{-1} \), the higher-order terms can safely be neglected and soliton propagation is governed by Eq. (3.18). However, for \( \omega_0 t_0 = \mathcal{O}(\varepsilon) \), the higher-order terms become important and solitons governed by the HNLS Eq. (3.19) are \textit{ultra-short}, of a width \( t_0 \sim \varepsilon \omega_0^{-1} \). These solitons are approximate solutions of Maxwell’s equations, satisfying Faraday’s and Ampère’s Laws in Eqs. (3.7) up to order \( \mathcal{O}(\varepsilon^3) \).

Finally, as concerns the condition for bright or dark soliton formation, by making a similar analysis with that of Eq. (3.18), we conclude that bright solitons are formed for \( \kappa < 0 \) and \( \sigma = -1 \) (i.e., \( \alpha < 0 \) with \( |\alpha / \beta| > Z_0^2 / Z_\perp^4 \)), while dark ones are formed for \( \kappa > 0 \) and \( \sigma = +1 \) (i.e., \( \alpha > 0 \), or \( \alpha < 0 \) with \( |\alpha / \beta| < Z_0^2 / Z_\perp^4 \)) [40]. These solutions are simply solitary waves, because the HNLS model is generally non-integrable (it is completely integrable only for specific values of its coefficients [36]). Nevertheless, results from direct numerical simulations reported in [42] and [43] show that these bright and dark solitary waves are quite robust during evolution.

4 Nonlinear Electromagnetic Bandgap (EBG) Metamaterials

4.1 Nonlinear Localization of Electromagnetic Waves in EBGs

In the two electromagnetic band-gaps (EBGs) of the considered Drude-Lorentz medium [c.f. Eqs.
(2.1) and (2.2), i.e. the low-frequency (LF) gap: \( 0 < \omega < \omega_{\text{res}} \), and the high-frequency (HF) gap: \( \omega_{M} < \omega < \omega_{\mu} \), linear waves are evanescent because in both frequency regions holds \( \Re\{\hat{\varepsilon}(\omega)\} < 0 \) and \( \Re\{\hat{\mu}(\omega)\} > 0 \) (see Fig. 1a). Still, if a nonlinearity occurs in the considered metamaterial then nonlinearity-induced localization of electromagnetic waves in the EBGs is possible. Such nonlinearities can appear e.g. in the dielectric permittivity function of the metamaterial (while the magnetic permeability is a linear function) [11],[28],[30]-[32],[44]-[46]. These assumptions are more realistic in the case of nonresonant, transmission-line metamaterials (cf., e.g., [47]-[49]; as was also demonstrated in experiments [50]-[53]). Note that the considered Drude-Lorentz form of the permittivity and permeability in the effective medium picture can be transformed to equivalent capacitances and inductors in the transmission-line description [54].

The localization of waves is indicated by the formation of electromagnetic gap solitons; such solitons have been predicted in nonlinear optics [55] and Bose-Einstein condensates (BECs) [56], by means of the NLS equation with a periodic potential. Gap solitons were also predicted to occur in nonlinear metamaterials [46] featuring a Drude-Lorentz frequency behavior. Precisely, by employing the slowly-varying electric and magnetic field envelopes approximation, a nonlinear Klein-Gordon (NKG) equation was obtained supporting gap solitons.

However, for the investigation of ultra-short pulse propagation, i.e, for pulse widths of the order of a few cycles of the carrier frequency, the NLS or the NKG models may fail. In the context of nonlinear fiber optics, the proper model describing the evolution of ultra-short pulses was shown to be the so-called short-pulse equation (SPE) [57]. Relevant numerical simulations comparing solutions of Maxwell’s equations to the ones of the SPE and NLS models have shown that the accuracy of the SPE (NLS) increases (decreases) as the pulse width shortens [57]. More recently, motivated by the fact that the SPE model has no smooth pulse solutions propagating with fixed shape and speed, a regularized SPE (RSPE) was derived [58]. The RSPE supports, under rather strict conditions, smooth traveling wave solutions.

### 4.2 Mathematical Modeling of Nonlinear Waves Propagation in the EBGs

We assume that the metamaterial exhibits a Kerr-type nonlinearity of the form (3.3) exclusively in its dielectric response. As in (3.5), we consider wave propagation along the \( z \)-direction of a \( x \)- (\( y \))-polarized electric (magnetic) field, and derive from Maxwell’s equations the following
time-domain nonlinear wave equation for the electric field $E(z,t)$

$$\partial_z^2 E - \partial_t^2 (\varepsilon F \mu F E) - \varepsilon_0 \hat{\alpha} \partial_t^2 (\mu F E) = 0,$$

(4.1)

where $\hat{\alpha}$ is the Kerr coefficient for the electric field; c.f. Eq. (3.3).

For the high-frequency (HF) EBG, defined by $\omega_m < \omega < \omega_p$ (see Fig. 1a), we assume that

$$\omega \gg \omega_{res}, \omega \gg \gamma_e, \text{ and } \omega \gg \gamma_\mu,$$

(4.2)

and, thus, obtain that $\hat{\varepsilon}_L(\omega)$ and $\hat{\mu}_L(\omega)$ in (2.1) and (2.2) are, respectively, approximated by

$$\hat{\varepsilon}_L(\omega) \approx \varepsilon_0 - \varepsilon_0 \frac{\omega^2}{\omega_0^2} + i \varepsilon_0 \frac{\gamma_e \omega_p^2}{\omega^3} \quad \text{and} \quad \hat{\mu}_L(\omega) \approx \mu_0 (1 - F) - \mu_0 F \frac{\omega^2}{\omega_0^2} + i \mu_0 F \frac{\gamma_\mu}{\omega}.$$

(4.3)

For typical parameter values, like the ones considered in [59], Eqs. (4.3) approximate with small relative errors the exact expressions (2.1) and (2.2) of the effective permittivity and permeability for a wide sub-interval of frequencies in the HF EBG (cf. Figs. 2 and 3 of [59]).

The terms of (4.1) involving convolution integrals are simplified by using (4.3), resulting in the following equation (in the frequency domain)

$$\partial_z^2 \hat{E} + \frac{1}{c^2} \left[ \omega^2 (1 - F) - (F \omega_p^2 + (1 - F) \omega_0^2) + i \omega F \gamma_\mu \right] \hat{E} + \frac{\hat{\alpha}}{c^2} \left[ \omega^2 (1 - F) - F \omega_{res}^2 + i \omega F \gamma_\mu \right] E^3 = 0,$$

(4.4)

where $\hat{E} = \int_{-\infty}^{\infty} E \exp(-i \omega t) dt$ is the Fourier transform of $E$ and $c$ the velocity of light in vacuum. In this way, we obtain the following time-domain nonlinear wave equation

$$\partial_t^2 E - \frac{1 - F}{c^2} \partial_z^2 E - \frac{1}{c^2} \left[ \frac{F \omega_p^2}{\omega_{res}} + (1 - F) \omega_0^2 \right] E - \frac{F \gamma_\mu}{c^2} \partial_t E$$

$$- \frac{\hat{\alpha}}{c^2} \left[ F \omega_{res}^2 E^3 + (1 - F) \partial_z^2 E^3 + F \gamma_\mu \partial_t E^3 \right] = 0.$$

(4.5)

Then, by measuring time, space, and the field intensity $E^2$ in units of $\omega_{res}^{-1}$, $c(1 - F)^{-1/2} \omega_{res}^{-1}$, and $|\hat{\alpha}|^{-1}$, respectively, Eq. (4.5) is cast in the dimensionless form

$$\left( \partial_z^2 - \partial_t^2 - \hat{\rho} - \gamma \hat{\alpha} \right) E = \frac{1}{1 - F} \left( \frac{F \partial_t^2 + \gamma \partial_t}{\omega_0^2} \right) E^3,$$

(4.6)

where $s = \text{sgn}(\alpha) = \pm 1$ for focusing or defocusing nonlinearity, respectively, while

$$\hat{\rho} = \frac{F}{1 - F} + \left( \frac{\omega_p}{\omega_{res}} \right)^2 \quad \text{and} \quad \gamma = \frac{F \gamma_\mu}{(1 - F)\omega_{res}}.$$

(4.7)
The validity of approximation (4.3) in a wide sub-interval of the HF EBG is assured by considering that \((\omega_p/\omega_{sc})^2 \gg 1\). In that case, \(\tilde{\rho}\) is a large parameter, which suggests that \(\tilde{\rho} = \rho/\varepsilon\), where \(\varepsilon\) is a formal small parameter, and \(\rho = \mathcal{O}(1)\).

In order to investigate the propagation of small-amplitude short pulses, we introduce a multiple scale ansatz of the form

\[
E = \varepsilon^{3/2}E_1(\tau, Z_1, \cdots) + \varepsilon^{3/2}E_2(\tau, Z_1, \cdots) + \cdots
\]

where

\[
\tau = (t - z)/\varepsilon \quad \text{and} \quad Z_n = \varepsilon^{n-1}z, \ n = 1, 2, \cdots
\]

Substituting Eq. (4.8) into Eq. (4.6), we find that terms at \(\mathcal{O}(\varepsilon^{-3/2})\) cancel, there are no terms at \(\mathcal{O}(\varepsilon^{-3/2})\), while terms at \(\mathcal{O}(\varepsilon^{-1/2})\), cancel provided that the field \(E_1 \equiv u\) satisfies the following equation

\[
2\partial_\zeta \partial_\xi u + \rho u + s\partial^3_\xi u^2 = -j \partial_\xi u,
\]

where \(\zeta = Z_1\). Equation (4.10) is a dissipative short pulse equation (DSPE), which was derived analytically in [21], and generalizes Eq. (14) of [59] that corresponded to a lossless metamaterial medium. Note that the lossless SPE \((\gamma = 0)\) was first derived in [57] as an appropriate model describing the propagation of ultra-short pulses in silica optical fibers with a Kerr nonlinearity.

### 4.3 Gap Solitons in EBG Metamaterials

The derived DSPE Eq. (4.10) may be used to model ultra-short pulses, that can emerge in the frequency band gap of the linear spectrum of the metamaterial. In this case, pulse-shaped solutions of the model are few-cycle pulses, characterized by widths of the order of only a few (e.g., 2-4) periods of the carrier wave [21].

In the absence of losses, the SPE model possess soliton solutions. In fact, the SPE is completely integrable [61], and its non-singular soliton solutions can be derived from the breather solutions of the sine-Gordon equation [62]. Apart from these exact solutions, approximate peakon-like and breather-like solitary waves, which can be regarded as ultrashort gap solitons in nonlinear metamaterials, were also presented [59] (see also [60]). Besides, localized and extended waveforms that arise in the SPE were considered in [63], and direct numerical simulations showed that the most robust solution is the breather type one (multi-peakon, multi-breather, and
multi-hump waveforms were found to be less robust during propagation). Furthermore, it is important to examine the effect of dissipation in the derived solutions. To this direction, one may employ techniques that have been established to investigate dynamics of solitons under perturbations, like, e.g., methodologies for perturbed NLS equations supporting bright soliton solutions [36], for perturbed NLS models possessing dark soliton solutions [64], as well as for perturbed SPE models [65]. Some initial conclusions on the dissipative dynamics of solitons associated to the DSPE model are as follows. From the DSPE Eq. (4.10), it is readily observed that, in the absence of the dispersive and nonlinear terms, the fields decay exponentially, that is, like \( \exp(-\gamma \zeta) \). Hence, the exponential decay rate, that is, the inverse of the linear loss coefficient, is determined explicitly in terms of the dielectric and magnetic properties of the Drude-Lorentz nonlinear metamaterial. Additionally, it is also worth to note that the solitons centers also evolve due to the presence of losses.

Analyzing systematically the dissipative dynamics of gap solitons solutions of the DSPE Eq. (4.10) is important not only from a theoretical but also from an experimental viewpoint in order to be able to properly predict and observe such soliton pulses in related applications. Equally important is the investigation of mechanisms that would sustain such coherent structures longer in terms of propagation distances in nonlinear metamaterials. In fact, compensating losses is a great challenge in the field of metamaterials because many interesting theoretical models were found (in the process of experimental realizations) to be suffering from significantly large losses. To this direction, many techniques have been proposed to overcome dissipation in metamaterials, e.g., embedding active circuits, using optical pumping techniques, injecting electrical current from semiconductors into large portions of metamaterials, using optical parametric amplification techniques, using noble conductors or even avoiding the use of metals, and utilizing electromagnetically induced transparency. Also, highly promising are configurations where metamaterials are combined with electrically and optically pumped gain media, such as semiconductor quantum dots, semiconductor quantum wells, and organic dyes embedded into the metal nanostructures, in order to achieve loss compensation from optical to terahertz spectral ranges. Overviews of related theoretical methodologies and experimental implementations are included in [66] and [67].

Finally, it is also worth to notice that higher-order corrections (in the form of, e.g., 4th-order derivative terms) have been proposed for the (lossless) SPE, giving rise to the so-called
regularized SPE (RSPE) [58]. Such corrections, play an important regularizing role on the SPE, which, otherwise, admits multi-valued (so-called “loop”) solutions [62]. The RSPE admits smooth traveling-wave solutions [58], which have similar structure to the NLS solitary waves [68]. However, in the Drude-Lorentz metamaterial under consideration such a 4th-order derivative term cannot be incorporated, due to the assumed form of the permittivity and permeability functions; see Eqs. (2.1) and (2.2). Considering a different functional form for \( \hat{\varepsilon}_L(\omega) \) and \( \hat{\mu}_L(\omega) \), e.g., the single-resonance double-Lorentz model (see, e.g., [69] as well as the recent work [70]), it could in principle be possible to derive such a dissipative RSPE model. Deriving such an RSPE model and analyzing its dissipative dynamics constitutes a particularly interesting future work direction.

Extensions of the SPE model to higher-dimensional settings are also expected to be of interest (this was investigated for 2+1 dimensions in [71]). To this end, relevant phenomena include solitons collapse, vortex formations, and so on.

5 Conclusions and Future Work Directions

In this paper, first, we presented an overview of the fundamental remarkable physical phenomena in (linear) left-handed electromagnetic metamaterials and listed the relevant experimental realizations. Then, we focused on nonlinear electromagnetic metamaterials whose effective dielectric permittivity \( \varepsilon \) and magnetic permeability \( \mu \) are functions of the fields intensities and hence can prove useful in tunable structures. Propagation of ultra-short electromagnetic pulses in lossy nonlinear metamaterials featuring a Drude-Lorentz behavior in \( \varepsilon \) and \( \mu \), respectively, was analyzed in detail. We started from Maxwell’s equations and employed two different perturbative approaches to derive two nonlinear partial differential equations governing electromagnetic wave propagation. The first equation (derived via the reductive perturbation method), is a higher-order nonlinear Schrödinger (NLS) model, while the second one is an short-pulse equation (SPE) model, both incorporating the effect of linear losses. The NLS and the SPE models describe ultra-short pulses which propagate in the left-handed regime and in the frequency band gaps of the considered metamaterial, respectively. Conditions for the formation of solitons solutions were obtained analytically, while the structure of these solutions was also discussed.

Interesting future work directions include systematic investigations of the stability of such ultra-short solitons both in the framework of the NLS or the SPE but also (and most importantly)
of the original Maxwell’s equations system. Higher-dimensional generalizations are also worth to be analyzed.

Finally, it is important to point out the recent great progress on nonlinear all-dielectric metamaterials and metasurfaces with applications, e.g., in shaping light [72], wavefront control, [73], nonlinear holography [74], nonlinear Fano-type resonances [75], and in generating nonlinear diffraction patterns [76], thus constituting the modeling of nonlinear dielectric metamaterials and the associated experimental implementations a very rich and active research field.

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Highlights

- Overview of remarkable effects and phenomena appearing in linear metamaterials
- Motivations and applications of nonlinear metamaterials
- Presentation of aspects on mathematical modeling and physical phenomena governing electromagnetic wave propagation in nonlinear metamaterials.
- In the left-handed band of the considered metamaterials, wave propagation is governed by a higher-order nonlinear Schrödinger (NLS) equation; ultra-short bright or dark solitons solutions are analytically derived.
- In the frequency band gaps, a dissipative short-pulse equation (DSPE) is derived. This equation governs ultrashort pulses that may be formed in the band gaps.