Numerical integration of orbits of planetary satellites

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Abstract. The 10th-order Gauss-Jackson backward difference numerical integration method and the Runge-Kutta-Nyström RKN12(10)17M method were applied to the equations of motion and variational equations of the Saturnian satellite system. We investigated the effect of step-size on the stability of the Gauss-Jackson method in the two distinct cases arising from the inclusion or exclusion of the corrector cycle in the integration of the variational equations. In the predictor-only case, we found that instability occurred when the step-size was greater than approximately 1/76 of the orbital period of the innermost satellite. In the predictor-corrector case, no such instability was observed, but larger step-sizes yield significant loss in accuracy. By contrast, the investigation of the Runge-Kutta-Nyström method showed that it allows_N seminar long step-sizes and can still obtain high-accuracy results, thus making evident the superiority of the method for the integration of planetary satellite systems.

Key words: methods: numerical – celestial mechanics – ephemerides – planets and satellites

1. Introduction

In this paper we aim to investigate how we can most efficiently apply common numerical integration methods to the dynamics of natural satellite systems. We consider the system which comprises the equations of motion of the Saturnian satellites together with the equations of the partial derivatives of the coordinates with respect to the parameters of the system (namely the initial conditions and the mass ratios of the satellites over Saturn together with the oblateness coefficients of Saturn). The values of the partial derivatives are required in order to fit the integration to observations or to another dynamical model. For the purpose of this integration we consider a rectangular Saturn-centric coordinate system in which Saturn’s equator plane is the xy-plane. This system is fully described in Sinclair & Taylor (1985).

The orbits of all the major Saturnian satellites are almost circular. Fox (1984) has shown that for near-circular problems the best method to use is the Gauss-Jackson method. As concerns the step-size that the Gauss-Jackson method should use, Herrick (1972) suggested that for the system of the equations of motion of a circular two-body problem, an appropriate step-size would be \( h = 1/(10n) \) where \( n \) is the mean motion of the orbiting body, and that in terms of the orbital period \( T \), this is \( h \approx T/63 \). Sinclair & Taylor (1985) used an 8th-order central-difference Gauss-Jackson method for the integration of the system of the three outer satellites Titan, Hyperion and Iapetus. They used predictor and corrector cycles for the equations of motion, but only a predictor cycle for the equations of the partial derivatives, as these are not needed to high accuracy. The integration covered a total time-span of 50 years and they used a step-size of 0.25 days, approximately 1/64 of the orbital period of Titan, the innermost satellite.

However, Shen (p.c.) discovered that when an additional satellite, Rhea, was introduced and the step-size reduced accordingly, an instability quickly arose in the integration of the equations for the partial derivatives.

In this work we first use a 10th-order backward-difference Gauss-Jackson method described by Herrick (1972) and Mer-son (1974). The latter presents a calculation of the coefficients of the method of order up to 15. In the present work we generate the coefficients of the 10th-order formulae using Maple (Char et al. 1991), and by repeating the numerical integration with various step-sizes we determine the largest step-size one could use in order to obtain results accurate enough (i) when only the predictor cycle is used for the integration of the partial derivatives, and (ii) when the corrector cycle is used as well. In the first case, a critical step-size is found, and attempts are made to analyse the behaviour of the method in the neighbourhood of this step-size.

In the second part of the paper, we apply to the same problem the most recent Runge-Kutta-Nyström (RKN) method, RKN12(10)17M of Dormand et al. (1987). Wu (p.c.) has shown that this is the most efficient of the RKN methods when high accuracies are required. The results of the two methods are compared and conclusions are made on the most efficient way to integrate the Saturnian satellite system.
### Table 1. Initial conditions and mass ratios

<table>
<thead>
<tr>
<th></th>
<th>Mimas</th>
<th>Dione</th>
<th>Titan</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position, in A.U.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_0$</td>
<td>0.000308890</td>
<td>-0.0024759195</td>
<td>-0.0079438545</td>
</tr>
<tr>
<td>$y_0$</td>
<td>0.0012611162</td>
<td>-0.0005107673</td>
<td>0.0002251206</td>
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<tr>
<td>$z_0$</td>
<td>-0.0000213951</td>
<td>0.0000001655</td>
<td>-0.0000197461</td>
</tr>
<tr>
<td>Velocity, in A.U. per day</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_0$</td>
<td>-0.0081202774</td>
<td>0.0011719753</td>
<td>-0.0001257187</td>
</tr>
<tr>
<td>$y_0$</td>
<td>0.0002785011</td>
<td>-0.0056597717</td>
<td>-0.0033045519</td>
</tr>
<tr>
<td>$z_0$</td>
<td>-0.0001168847</td>
<td>-0.0000008699</td>
<td>0.0000183595</td>
</tr>
<tr>
<td>Mass ratios, satellite/Saturn</td>
<td>6.46 $\times$ 10$^{-8}$</td>
<td>1.916 $\times$ 10$^{-6}$</td>
<td>2.3678 $\times$ 10$^{-4}$</td>
</tr>
</tbody>
</table>

2. The Gauss-Jackson method

For the numerical investigation of the Gauss-Jackson method we used a program which integrates the equations of motion and the equations of the partial derivatives of $n$ bodies using a 10th-order backward-difference Gaussian formula. As the choice of the step-size depends on the period of the innermost satellite only, we simplified the program so that it integrates only one satellite. Therefore there are no mutual satellite perturbations to be taken into account in the force model. For simplification, we also neglect solar and oblateness perturbations, as their contribution to the accelerations is very small compared with that of the central force. This two-body approach is not expected to prevent a generalisation of our results as, in the absence of close commensurabilities, the main effect of mutual and oblateness perturbations upon the partial derivatives is only the “constant perturbation”, i.e. a slight change in the satellite’s apparent mean motion.

We chose to integrate the orbit of Mimas, the innermost satellite, which has an orbital period of $T = 0.9424219$ days. The initial conditions for Mimas and the mass-ratio Mimas/Saturn are given in Table 1.

As noted above, the inclusion of the equations of the partial derivatives in the system was found to cause instability at large step-sizes. It was therefore considered necessary to investigate the exact conditions under which the instability occurs. It is easy to show that, for a near circular orbit, the partial derivatives are bounded by the relations:

$$\left| \frac{\partial x}{\partial \xi_0} \right| \lesssim 3nt$$  \hspace{1cm} (1)

and

$$\left| \frac{\partial x}{\partial \xi_0} \right| \lesssim 3t$$  \hspace{1cm} (2)

where $x$ is any coordinate of the satellite, $\xi_0, \dot{\xi}_0$ are any components of the satellite’s initial position and velocity respectively, $n$ is the satellite’s mean motion, and $t$ the time since the start of the integration. Therefore, it is possible to monitor the behaviour of the partials by comparing them at each step with these bounds. In fact, the orbit of Mimas is not exactly circular but has an eccentricity of order 0.01, so the above bounds are only approximate, and it is necessary to increase them slightly, say by a multiple of 1.1, in order to ensure that, when the partials are well-behaved, they remain within the bounds.

This monitoring procedure was included in the program in such a way that at each time-step, the absolute values of all the partial derivatives were compared to their theoretical bounds (3$nt$ or 3$t$ accordingly, multiplied by 1.1), and the integration was halted as soon as any of the partial derivatives exceeded its bound. Subsequently two distinct cases were studied: the case in which only the predictor cycle was used for the integration of the partial derivatives, and the case in which a corrector cycle was also included.

2.1. Study of the predictor-only case

We initially tried $T/64 \approx 0.0147$ d as a step-size. The partials exceeded the bounds very quickly, in only 861 steps. The experiment was repeated by gradually decreasing the step-size. After many trials, it was found that the critical step-size lies between $T/75$ and $T/76$.

Running the integration with the larger step-size showed rapid exponential growth of all the partials after about 25000 steps, while using the smaller step-size and integrating the system for a time-span of 2 million steps (almost 68 years), showed that all of the partial derivatives stayed within their theoretical bounds throughout the whole integration.

In order to see if the results obtained above are general, that is they are independent of the choice of the satellite, we repeated the same experiment using Dione, whose orbital period is $T_d = 2.736915548$ days. Again, with the step-size of $T_d/75$ the partials started exceeding their bounds at 13000 steps, while with $T_d/76$ all the partials were kept within the bounds even after 2 million steps (roughly 197 years). We also tried integrating Titan, which has an orbital period $T_t = 15.945448$ days. Because Titan’s mean motion is quite small, a multiple of 2.0 was used for the bounds. The results were exactly the same as for the other two satellites. Thus the limit $h = T/76$ applies to all satellites, independently of the actual orbital period. The initial conditions adopted for Dione and Titan are listed in Table 1.

Throughout the remainder of this paper, step-sizes will be quoted in terms of the orbital period of Mimas, which was used as our test object.

In order to check the accuracy of the coordinates obtained with the step-size of $h = T/76$, we checked the value of a constant of motion, the energy integral:

$$\frac{\mu^2}{r} - \frac{2 \mu}{r} = -\frac{1}{\mu},$$  \hspace{1cm} (3)

where $r,v$ are the magnitudes of the position and velocity vectors, $a$ is the semi-major axis of the orbit and $\mu = GM(M + m)$, where $G$ is the gravitational constant, $M$ is the mass of the
planet, and $m$ is the mass of the satellite. The correct value of
the integral is considered to be the one calculated from the initial
conditions.

After 1 million steps the absolute error in the energy integral
reached the value of $6.27 \times 10^{-9}$. This is sufficiently small to
allow us to regard the integration of satellite’s coordinates to be
accurate, and thus we can conclude that $h = T/76$ is the best
step-size one could use for the integration of the coordinates and
the partial derivatives with Gauss-Jackson backward-difference
method when only the predictor cycle is used for the partial
derivatives.

The fact that Sinclair & Taylor (1985) used a step-size of $1/64$ of
Titan’s orbital period with an 8th-order central-difference
formula can be explained by the more rapid decrease of the
coefficients of the central-difference formulae compared to
those of the backward-difference formula that we have been
using. This tends to reduce the growth of error in the central-
difference method and this is the reason why Sinclair & Taylor
could use a larger step-size even though the formula was of lower
order. However, the backward-difference method is much sim-
pler to program and somewhat more efficient in execution. This
more than offsets the slight decrease in speed due to the smaller
step-size compared to the central-difference method.

As a first attempt to find the reasons for instability in the
partial derivatives when a step-size larger than $T/76$ is used, we
examined whether the instability was in any way related to the
order of summations in the backward-difference formulae. Ac-
ccording to Milani & Nobili (1988), the main source of rounding
error in a numerical integration method lies in the summation
of the small quantity which represents the change in position at
each step, compared to the previous position, which is a much
larger quantity. In the formulae of a backward-difference Gauss-
Jackson method the smallest quantity in every summation is the
highest-order difference. Therefore, in our case, summations
should be done starting from the $10^{th}$-order differences and
in decreasing order. After applying this rule to our program,
we found exactly the same results as before. Thus the order of
summation does not influence the accuracy of the backward-
difference Gauss-Jackson method.

As another approach to the problem, we checked the be-
behaviour of the $10^{th}$-order differences, as any possible instability
should first become apparent in the highest-order differences
of a Gauss-Jackson scheme. For a circular orbit, the $10^{th}$-order
differences are bounded by the relationship

$$|\nabla^{10} \mathbf{x}| \leq 3nt(hn)^2.$$  \hspace{1cm} (4)

It was found that for all of the partial derivatives, the ampli-
tude of the differences increased exponentially with time when
a step-size of $T/75$ was used and linearly with a step-size of
$T/76$. Therefore it was considered necessary to investigate fur-
ther what exactly is happening as we slightly decrease the step-
size from $T/75$ to $T/76$.

For this purpose we modified the program so that it per-
fomed two integrations at the same time, one with a chosen
test step-size and one with half the test step-size. The latter
was assumed to give the correct answers, therefore it was used
for comparison with the results of the integration with the test
step-size. As a result, we could plot the relative error in the par-
tial derivatives between the two integrations, and see how this
evolves with time.

We ran this program successively using the following test
step-sizes: $T/75.1, T/75.2, T/75.3, T/75.4$. The last of these
values showed good behaviour of the partial derivatives, and
therefore we did not use a smaller test step-size. The results
for partial $\partial \mathbf{x}/\partial \mathbf{x}_0$ are shown in Table 2, and are completely
analogous to the results for the rest of the partials.

Decreasing the step-size to $T/75.4$ and integrating for 2 mil-
ion steps (a time-span of 68 years, which is definitely ade-
quate for any proposed numerical integration of the Saturnian
satellites), we find that the exponential behaviour is not evident (or
at least it must start beyond 2 million steps). The relative error
in all partials at the end of the integration is of order $10^{-9}$ and
therefore the accuracy is adequate.

### 2.2. Study of the predictor-corrector case

The introduction of the corrector cycle in the integration of the
partial derivatives changed the above results considerably: not
only did the partial derivatives remain within their theoretical
bounds with a step-size of $T/64$, but also with a step-size of
$T/32$. However, before reaching conclusions about the suit-
ability of these step-sizes, we must first check the accuracy of
the above results. This was done in two ways:

1. We integrated Mimas with 3 different step-sizes: $T/32,$
$T/64$ and $T/128$ for 1, 2 and 4 million steps respectively,
i.e. for a time-span of 80 years. Then, assuming that the
third integration ($h = T/128$) gives correct results, we stud-
ied the relative errors in the partial derivatives between the
third integration and the other two. After 2 million steps the
relative error for $T/64$ remains small (of order $10^{-3}$),
while this is not the case with the $T/32$ integration (the maximum
relative error is of order 1), and therefore we are obliged to
conclude that a step-size of $T/32$ is not appropriate.

2. We again checked the error in the energy integral when inte-
lgrating with step-size $T/64$ for 840000 steps and with $T/32$
for 420000 steps, so that the time-span of all the integrations
is the same (about 34 years). After this time period, the ab-
solute error in the $T/64$ integration had reached the value of
$6 \times 10^{-8}$ while in the $T/32$ integration it was as large as
$1.4 \times 10^{-4}$.

From these experiments, we can conclude that the predictor-
corrector method may enable us to use much larger step-sizes

### Table 2. Results for partial $\partial \mathbf{x}/\partial \mathbf{x}_0$ in the predictor-only case

<table>
<thead>
<tr>
<th>Step-size</th>
<th>Onset of instability (steps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T/75.1$</td>
<td>10000</td>
</tr>
<tr>
<td>$T/75.2$</td>
<td>20000</td>
</tr>
<tr>
<td>$T/75.3$</td>
<td>130000</td>
</tr>
<tr>
<td>$T/75.4$</td>
<td>Stable</td>
</tr>
</tbody>
</table>
than the predictor-only method, but this is not of much use, as a slight increase in the step-size leads to significant loss of accuracy in the coordinates and the partial derivatives.

Nevertheless, one may ask whether it is worth using the predictor-corrector method with a step-size as small as \( T/76 \), because this should give more accurate values of the partial derivatives than the predictor-only method. To answer this question, we compared the values of the partial derivatives taken from the integrations with both methods, using the same step-size \( T/76 \) and covering 1 million steps. After this period of time, the results of the two integrations were still the same to 5 significant digits. This implies that if we are satisfied with a relative accuracy of \( 10^{-5} \) for the partial derivatives (which is usually the case) it is adequate to use the Gauss-Jackson with the predictor-only cycle for the integration of the partial derivatives.

However, we may draw one more conclusion: when the corrector cycle is included in the integration of the partial derivatives, the step-size can exceed the critical value of \( T/76 \) established in experiments with the predictor-only method. This proves that the exponential growth of the error in the partial derivatives that occurred in the predictor-only method was due to an instability of the Gauss-Jackson method itself, and is not implicit in the system of equations.

3. The Runge-Kutta-Nyström method

We have also used the NAG library’s implementation of the RKN12(10)17M Runge-Kutta-Nyström method (Brankin et al. 1987, 1989) to integrate the Mimas system. This method is characterised by adaptive step-size control in which the user may set limits on the permissible local error at each step of the integration.

We performed various trials with different tolerances of the local relative error of the coordinates and the partial derivatives. The local error limit was the only parameter of the routine that we controlled directly. In fact, in order to allow the method to choose step-sizes as large as possible, we used the code in interval mode where each interval was approximately 1.3\( T \). For comparison with the results of the Gauss-Jackson predictor-only method, the integration was carried out over a total of 10000 such intervals, thus covering the same time-span as the trials with the Gauss-Jackson method (34 years).

We started the trials with a tolerance of \( 10^{-10} \). The step-sizes used by the routine ranged from \( T/30 \) to \( T/9.5 \). Comparing the results obtained with those of the Gauss-Jackson predictor-only method with step-size \( T/76 \), we found that the coordinates and the partial derivatives of the two integrations agreed to about 7 significant digits at the start of the integration and to 4 significant digits at the end. In order to find out which of the two methods was more accurate, we checked the RKN by monitoring the energy integral, and found that the absolute error had reached the value of \( 8.86 \times 10^{-8} \) at the end of the integration. Considering that the error in Gauss-Jackson of step-size \( T/76 \) was \( 6.27 \times 10^{-9} \), we conclude that the latter is more accurate than the RKN with a tolerance of \( 10^{-10} \). However, the RKN is using much larger step-sizes, and thus needs only 140127 steps to cover the time-span of 34 years, instead of 1 million steps needed by the Gauss-Jackson.

Decreasing the tolerance to \( 10^{-11} \), we found that the error in the energy integral became \( 4.7 \times 10^{-9} \), i.e. it is a little better than the error of Gauss-Jackson. The number of steps needed for the integration was only 170258, and the step-sizes used were in the range of \( T/40 \) to \( T/11.6 \). However, before concluding the superiority of the RKN12(10)17M against the 10th-order Gauss-Jackson, we did one more comparison, this time measuring the CPU-time needed for each method to cover the same time-span. On a VAX 9000 machine, the Gauss-Jackson method took about 9 minutes to integrate the system for the time-span of 34 years, while the RKN of tolerance \( 10^{-11} \) took about 11 minutes for the same time-span. Despite using much larger step-sizes than the Gauss-Jackson method, the RKN method needs slightly more CPU-time since it performs 17 function evaluations at each step instead of only two performed by the Gauss-Jackson. However, for comparing the two methods in terms of rapidity, more complex test problems should be used (Dormand et al. 1987).

We continued to decrease the tolerance of the RKN in order to determine the tolerance which achieves the best accuracy. At a tolerance \( 10^{-12} \), the error at the end of the integration decreased further to \( 1.3 \times 10^{-9} \). The step-sizes needed were still relatively large (from \( T/39.45 \) to \( T/14.8 \)) and the total number of steps was 210457. This accuracy is the best that can be attained for this specific system, because for tolerances of \( 10^{-13} \) and smaller, the error actually starts increasing slightly. This is due to the accumulation of the global error, because of the increase of the number of steps needed to cover the time-span.

4. Computational considerations

The numerical experiments described above were initially done on a DRS6000 machine of Queen Mary and Westfield College, and then continued on the VAX 9000 of the University of Thessaloniki. Despite the slightly different representation of double-precision numbers on the VAX (mainly related to the different number of significant bits in the mantissa) and also the possible use of different rounding techniques between the two machines, there was no effect at all on the results of the numerical investigation. This demonstrates that the Gauss-Jackson instability is independent of the representation of floating-point numbers.

5. Conclusions

The main objective of this paper was the investigation and comparison of two numerical integration methods: the 10th-order backward-difference Gauss-Jackson formula and the 12(10)17 Runge-Kutta-Nyström, when applied to the system of equations of motion and differential equations of the partial derivatives of the system of Saturnian satellites. The use of the Gauss-Jackson method with the predictor only cycle for the integration of the partial derivatives showed an instability of the method, which does not allow the use of step-sizes larger than 1/76 of the orbital...
period of the innermost satellite. Using this optimal step-size and integrating Mimas for 1 million steps, we achieved a final relative error in the energy integral of order $10^{-11}$, and therefore a good accuracy in coordinates as well as in the partial derivatives.

Furthermore, when we introduced the corrector cycle for the integration of the partials, the instability disappeared and we were then able to use much larger step-sizes, but at the cost of significant loss in accuracy of the results obtained: for step-size $T/64$, the relative error in the energy integral was of order $10^{-10}$ and the relative error in partial derivatives was of order $10^{-3}$, while for step-size $T/32$, the relative errors were respectively of orders $10^{-6}$ and $10^{-1}$. Even with step-sizes as small as $T/76$, the increase in accuracy of the partial derivatives obtained by the inclusion of the corrector cycle is trivial compared to the increase in integration-time, and therefore the predictor-only cycle should be preferred.

By contrast, the RKN12(10)17M method can achieve very good accuracy with step-sizes much larger than $1/76$ of the period, and therefore it is the most appropriate method for the integration of this system of equations. Using a tolerance of $10^{-12}$ and integrating Mimas for the same time-span, the stepsizes used were as large as $1/40$ to $1/15$ of the satellite's orbital period, and the relative error in the energy integral reached the value of $10^{-11}$. With this tolerance we can best exploit the method's performance.

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