Additive and Multiplicative Hankel Operators

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The thesis consists of two parts, the theory of additive Hankel operators on the Hardy space $H^2$ and the theory of multiplicative Hankel operators on the Hardy space of Dirichlet series $H^2$. Hankel introduced matrices with entries that depend only on the sum of the coordinates (Hankel matrices), these matrices induce the additive Hankel operators. On the other hand, it was Helson who introduced multiplicative Hankel operators or Helson matrices (the entries depend only on the product of the coordinates). The main aim of this thesis is to investigate the existence or absence of characterizations for bounded, compact and Schatten class Hankel operators (additive and multiplicative).

The first part begins with some preliminaries, we study briefly the theory of compact operators on separable Hilbert spaces, we define the Schatten classes and we continue making a short presentation of analytic function spaces such as Hardy, BMO, Bergman, Bloch and Besov spaces. In the second chapter, we define the additive Hankel operators and we prove with all the details the Nehari theorem (characterization of boundedness), Hartman’s theorem (characterization of compactness) and Peller’s theorem (characterization of Schatten class Hankel operators).

The second part starts with the basic theory of Dirichlet series and the theory of Hardy Spaces of Dirichlet Series, which suggested by Beurling in 1945 and introduced by H. Hedenmalm, P. Lindqvist, K. Seip in 1997 when they solved the Riesz basis problem. We focus on a correspondence (Bohr-lift) between those spaces and the Hardy spaces in infinitely many variables. In the last chapter, we define the multiplicative Hankel operators, we study their properties and we prove that the analogues of the previous characterizations fail.
<table>
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<tr>
<th>Notation</th>
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<tr>
<td>$\hat{f}(n)$</td>
<td>The Fourier coefficient of the integrable function $f$.</td>
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<tr>
<td>$\mathbb{D}$</td>
<td>The unit disk.</td>
</tr>
<tr>
<td>$\mathbb{T}$</td>
<td>The unit circle.</td>
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<tr>
<td>$H(\Omega)$</td>
<td>The set of all holomorphic functions on the open set $\Omega \subset \mathbb{C}$.</td>
</tr>
<tr>
<td>$f_r$</td>
<td>The function $f_r(z) = f(rz)$.</td>
</tr>
<tr>
<td>$|\cdot|$</td>
<td>The norm defined on a linear space.</td>
</tr>
<tr>
<td>$\langle \cdot, \cdot \rangle$</td>
<td>The inner product (if exists) of a complex vector space.</td>
</tr>
<tr>
<td>$B_X(x_0, r)$</td>
<td>The ball in $(X, |\cdot|_X)$ with center $x_0$ and radius $r &gt; 0$.</td>
</tr>
<tr>
<td>$m$</td>
<td>The normalized Lebesgue measure of $\mathbb{T}$.</td>
</tr>
<tr>
<td>$L^p(X, d\mu)$</td>
<td>The standard $L^p$ space of measurable functions on $X$ with respect to the measure $\mu$.</td>
</tr>
<tr>
<td>$\ell^p$</td>
<td>The normed space of all complex sequences $a = {a_n}<em>{n \geq 1}$ with finite $p$–norm $|a|</em>{\ell^p} := \left( \sum_{n \geq 1}</td>
</tr>
<tr>
<td>$\ell_\infty$</td>
<td>The normed space of all bounded complex sequences with respect to the supremum norm.</td>
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<tr>
<td>$c_\infty$</td>
<td>The space of all finitely supported complex sequences.</td>
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Part I

Additive Hankel Operators
Chapter 1

Preliminaries

In this Thesis we shall assume that the reader has the standard knowledge on measure theory and integration, functional analysis, complex analysis and the theory of Hardy spaces of the unit disk.

1.1 Operator Theory

1.1.1 Compact Operators

Definition 1.1.1. Let $X$ be a vector space over $\mathbb{C}$, a norm on $X$ is a real-valued function $\|\cdot\| : X \rightarrow [0, +\infty)$ that satisfies the following properties:

1. $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$.
2. $\|\lambda x\| = |\lambda| \|x\|$, $\forall x \in X$, $\forall \lambda \in \mathbb{C}$.
3. $\|x\| = 0$ if and only if $x = 0$.

The space $X$ equipped with the norm $\|\cdot\|$ is a normed space and it is a metric space with respect to the induced metric $d(x, y) = \|x - y\|$. A Banach space is a complete normed space.

An inner product space $H$ is a vector space equipped with a function $\langle \cdot, \cdot \rangle : H \rightarrow \mathbb{C}$, called inner product, that satisfies the following conditions:

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\forall x, y \in H$.
2. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, $\forall x, y, z \in H$.
3. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$, $\forall x, y \in H$, $\forall \lambda \in \mathbb{C}$.
4. $\langle x, x \rangle \geq 0$, $\forall x \in H$ and $\langle x, x \rangle = 0$ if and only if $x = 0$. 


The space $H$ is a normed space with respect to the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$. A Hilbert space is a complete inner product space.

A linear operator $T$ between two Banach spaces $X, Y$ is bounded if there exists a constant $C > 0$ such that

$$\|T(x)\| \leq C \|x\|, \forall x \in X.$$  

The infimum of these constants $C$ is the norm of the operator $T$

$$\|T\| = \inf\{C > 0 : \|T(x)\| \leq C \|x\|, \forall x \in X\}.$$  

The space $B(X, Y)$ of bounded linear operators between the Banach spaces $X, Y$ is a Banach space with respect to the operator-norm. The operator $T$ is compact if it maps every bounded set to a relatively compact one.

For our purpose it is sufficient to work on separable Hilbert spaces.

**Proposition 1.1.1.** Let $H$ be a Hilbert spaces and $T : H \rightarrow H$ be a bounded linear operator. $T$ is compact if and only if there exists a sequence of bounded operators of finite rank, i.e. the dimension of the image is finite, $\{S_n\}_{n \geq 1}$ such that

$$\|S_n - T\| \rightarrow 0.$$  

**Proof.** A set $S$ in a finite dimensional normed space $X$ is compact if and only if it is closed and bounded, so every bounded operator of finite rank is compact.

Suppose $T$ is compact and let $\{e_1, \ldots, e_n, \ldots\}$ be an orthonormal basis for $H$. We define the sequence of operators $\{I_n\}_{n \geq 1}$ on $H$ as

$$I_n(x) = \sum_{j=1}^{n} \langle x, e_j \rangle e_j, \ n = 1, 2, \ldots.$$  

By Parseval’s identity, $\{I_n\}_{n \geq 1}$ is a sequence of contractions that converges pointwise to the identity operator. Contractivity implies equicontinuity and as a consequence $\{I_n\}_{n \geq 1}$ converges uniformly to the identity operator $I : H \rightarrow H$ on every compact set. Thus, on the compact set $T\overline{B_H(0,1)}$, we have that

$$\|I_n \circ T - T\| = \|I_n \circ T - I \circ T\| = \sup\left\{ \|I_n(x) - I(x)\|_H : x \in T\overline{B_H(0,1)} \right\} \rightarrow 0.$$  

We conclude that $\{I_n \circ T\}_{n \geq 1}$ is a sequence of bounded operators of finite rank such that

$$\|I_n \circ T - T\| \rightarrow 0.$$
Conversely, suppose that there exists a sequence of bounded operators of finite rank \( \{S_n\}_{n \geq 1} \) such that
\[
\|S_n - T\| \to 0.
\]
It is easy to prove that a set \( S \) in a complete metric space \( X \) is relatively compact if and only if for every \( \epsilon > 0 \) there exists a finite sequence of points \( x_1, \ldots, x_n \in X \) such that
\[
S \subseteq \bigcup_{j=1}^{n} B_X(x_j, \epsilon).
\]
Let \( \epsilon > 0 \), there exist \( n_0 \in \mathbb{N} \) and \( x_1, \ldots, x_n \in H \) such that
\[
\|S_{n_0} - T\| < \frac{\epsilon}{2} \quad \text{and} \quad S_{n_0}(B_H(0,1)) \subseteq \bigcup_{j=1}^{n} B_H(x_j, \frac{\epsilon}{2}).
\]
This implies that
\[
T(B_H(0,1)) \subseteq \bigcup_{j=1}^{n} B_H(x_j, \epsilon).
\]
\[\blacksquare\]

Now we will define the Adjoint operator of a bounded linear operator \( T \) on the separable Hilbert space \( H \). By the Hahn-Banach and the Riesz representation theorem it is easy to prove that
\[
\|x\| = \sup \{|\langle x, y \rangle| : \|y\| = 1\}, \quad \forall x \in H.
\]
For an arbitrary \( y \in H \) the linear functional \( H \ni x \mapsto \langle T(x), y \rangle \) is bounded and by the Riesz representation theorem there exists a vector \( T^*(y) \in H \) such that
\[
\langle T(x), y \rangle = \langle x, T^*(y) \rangle, \quad x, \ y \in H.
\]
The adjoint operator is defined by
\[
T^* : H \to H, \quad y \mapsto T^*(y).
\]
It is easy to prove that \( T^* \) is a linear operator with norm \( \|T^*\| = \|T\| \).

**Example.** The Shift operator \( S : H^2 \to H^2 \) is defined as
\[
S(f) = zf(z), \quad f \in H^2, \quad z \in \mathbb{D},
\]
or equivalently
\[
S \left( \sum_{n \geq 0} \hat{f}(n) z^n \right) = \sum_{n \geq 0} \hat{f}(n) z^{n+1}.
\]
\( S \) is a contraction and the adjoint operator \( S^* : H^2 \to H^2 \) called the Backward Shift operator
\[
S^* \left( \sum_{n \geq 0} \hat{f}(n) z^n \right) = \sum_{n \geq 0} \hat{f}(n+1) z^n.
\]
Proposition 1.1.2. Let $T$ be a bounded operator of finite rank on $H$, then $T^*$ is also a bounded operator of finite rank.

Proof. Every operator $T$ of finite rank is a finite sum of rank-one operators and the action "*' is additive. It is sufficient to prove that the adjoint operator of a rank-one bounded operator $T$, is also a rank-one operator.

Let $w \in H$, we consider the rank-one operator

$$T(x) = l_x w, \quad x \in H.$$ 

The coefficient $l_x$ is a bounded linear functional, by the Riesz representation theorem there exists a $z \in H$ such that

$$T(x) = \langle x, z \rangle w.$$ 

We observe that for every $x, y \in H$

$$\langle x, T^*(y) \rangle = \langle T(x), y \rangle = \langle x, z \rangle \langle w, y \rangle = \langle x, \langle y, w \rangle z \rangle.$$ 

It follows that

$$T^*(y) = \langle y, w \rangle z, \quad y \in H.$$ 

Proposition 1.1.3. Let $T$ be a compact operator on $H$, then $T^*$ is also compact.

Proof. There exists a sequence of bounded operators of finite rank $\{S_n\}_{n \geq 1}$ such that

$$\|S_n - T\| \to 0.$$ 

$\{S_n^*\}_{n \geq 1}$ is also a sequence of bounded operators of finite rank and

$$\|S_n^* - T^*\| = \|(S_n - T)^*\| = \|S_n - T\| \to 0.$$ 

Definition 1.1.2. Let $H$ be a Hilbert space. A sequence $\{x_n\}_{n \geq 1} \subseteq H$ converges weakly to $x \in H$ if $\langle x_n, y \rangle \to \langle x, y \rangle$, for every $y \in H$.

Proposition 1.1.4. For every bounded sequence $\{x_n\}_{n \geq 1} \subseteq H$, there exists a weakly convergent subsequence $\{x_{n_k}\}_{k \geq 1}$.

Proof. Let $M < +\infty$ such that $M > \|x_n\|_H$, $\forall n \in \mathbb{N}$ and let $\{y_1, y_2, \ldots\}$ be a dense subset of $H$. $\{(y_1, x_n)\}_{n \geq 1}$ is a bounded sequence in $\mathbb{C}$, so there exists a convergent subsequence $\{(y_1, x_{n_{k_1}})\}_{k_1 \geq 1}$ and similarly $\{(y_2, x_{n_{k_1}})\}_{k_1 \geq 1}$ is a bounded sequence in $\mathbb{C}$, so there exists a convergent subsequence $\{(y_2, x_{n_{k_2}})\}_{k_2 \geq 1}$, we continue in this
way and by a diagonal argument we can find a subsequence \( \{x_{n_k}\}_{k \geq 1} \) such that \( \langle y_m, x_{n_k} \rangle_{k \geq 1} \) converges for every \( m \in \mathbb{N} \). Let an arbitrary \( y \in H \), we will prove that \( \langle y, x_{n_k} \rangle_{k \geq 1} \) converges to a limit \( l_y \). For every \( \epsilon > 0 \) there exists a vector \( y_{n_0} \) such that

\[
\|y - y_{n_0}\|_H \leq \frac{\epsilon}{3M}. \tag{1.1}
\]

Since \( \{\langle y_{n_0}, x_{n_k} \rangle\}_{k \geq 1} \) converges, there exists a \( N \in \mathbb{N} \) such that

\[
|\langle y_{n_0}, x_{n_p} \rangle - \langle y_{n_0}, x_{n_q} \rangle| \leq \frac{\epsilon}{3}, \quad p, q \geq N. \tag{1.2}
\]

By (1.1) and (1.2)

\[
|\langle y, x_{n_p} \rangle - \langle y, x_{n_q} \rangle| \leq \epsilon, \quad p, q \geq N.
\]

This implies that \( \{\langle y, x_{n_k} \rangle\}_{k \geq 1} \) converges to a limit \( l_y \). It is easy to prove that \( y \mapsto l_y \) is a linear functional and that \( |l_y| \leq M \|y\|_H \). By the Riesz representation theorem there exists a \( x \in H \) such that

\[
l_y = \langle y, x \rangle = \lim_{k \to \infty} \langle y, x_{n_k} \rangle.
\]

So, \( \{x_{n_k}\}_{k \geq 1} \) converges weakly to \( x \).

A direct consequence of the uniform boundedness principle is that every weakly convergent sequence is bounded.

**Proposition 1.1.5.** Let \( T \) be a bounded operator on \( H \). Then, \( T \) is compact if and only if for every sequence \( \{x_n\}_{n \geq 1} \subseteq H \) that converges weakly to 0,

\[
\|T(x_n)\|_H \to 0.
\]

**Proof.** Suppose \( T \) is compact and let \( \{x_n\}_{n \geq 1} \) be a bounded sequence that converges weakly to 0. There exists a subsequence \( \{T(x_{n_k})\}_{k \geq 1} \) that converges to a vector \( y \in H \). It is sufficient to prove that 0 is the unique limit point of \( \{T(x_n)\}_{n \geq 1} \). Let \( w \) be an arbitrary vector in \( H \)

\[
\langle y, w \rangle = \lim_{k \to \infty} \langle T(x_{n_k}), w \rangle = \lim_{k \to \infty} \langle x_{n_k}, T^*(w) \rangle = 0.
\]

So, 0 is the unique limit point.

Conversely, suppose that for every sequence \( \{x_n\}_{n \geq 1} \subseteq H \) that converges weakly to 0 holds that

\[
\|T(x_n)\|_H \to 0.
\]

It is sufficient to prove that for every bounded sequence \( \{x_n\}_{n \geq 1} \) the sequence \( \{T(x_n)\}_{n \geq 1} \) has a convergence subsequence. Since \( \{x_n\}_{n \geq 1} \) is bounded, there exists a \( x \in H \) and a subsequence \( \{x_{n_k}\}_{k \geq 1} \) such that

\[
x_{n_k} - x \to 0, \quad \text{weakly}.
\]

So, \( \{T(x_{n_k})\}_{k \geq 1} \) converges to \( T(x) \).
Theorem 1.1.1. A sequence \( \{f_n\}_{n \geq 1} \subseteq H^2 \) converges weakly to 0 if and only if \( \{f_n\}_{n \geq 1} \) is bounded in \( H^2 \) and converges locally uniformly to 0.

**Proof.** Suppose \( \{f_n\}_{n \geq 1} \subseteq H^2 \) converges weakly to 0, then \( \{f_n\}_{n \geq 1} \) is bounded in \( H^2 \) and as a consequence is locally uniformly bounded. Applying Montel’s Theorem we can find a subsequence \( \{f_{n_k}\}_{k \geq 1} \) that converges locally uniformly to a function \( g \in H^2 \). It is sufficient to prove that 0 is the unique local uniform limit point of \( \{f_n\}_{n \geq 1} \). Let \( \{e_1, e_2, \ldots\} \) be the standard orthonormal basis of \( H^2 \), then

\[
\lim_{k \to \infty} \langle f_{n_k}, e_m \rangle = 0, \quad \forall m \in \mathbb{N}.
\]

Thus,

\[
\lim_{k \to \infty} \frac{f_{n_k}^{(m)}(0)}{m!} = 0 = \frac{g^{(m)}(0)}{m!}, \quad \forall m \in \mathbb{N} \cup \{0\}.
\]

So, \( g \equiv 0 \).

Conversely, suppose that \( \{f_n\}_{n \geq 1} \) is bounded in \( H^2 \) and locally uniformly convergent to 0. There exists a subsequence \( \{f_{n_k}\}_{k \geq 1} \) that converges weakly to a function \( g \in H^2 \). It is sufficient to prove that 0 is the unique weak accumulation point of \( \{f_n\}_{n \geq 1} \).

\[
0 = \lim_{k \to \infty} \frac{f_{n_k}^{(m)}(0)}{m!}, \quad \forall m \in \mathbb{N}
\]

\[
0 = \lim_{k \to \infty} \langle f_{n_k}, e_m \rangle, \quad \forall m \in \mathbb{N}
\]

\[
0 = \langle g, e_m \rangle, \quad \forall m \in \mathbb{N}.
\]

We conclude that \( g \equiv 0 \). □

The above theorem remains true if we replace \( H^2 \) with any weighted Bergman space \( A^2_{a}, \ a > -1 \), which we will study later in this chapter.

**Proposition 1.1.6.** Let \( \{\sigma_n\}_{n \geq 1}, \ \{e_n\}_{n \geq 1} \) be two orthonormal sequences in \( H \) and let \( a = \{a_n\}_{n \geq 1} \) be a sequence of complex numbers that converges to 0. Then, the operator \( T : H \to H \) defined as \( T(x) = \sum_{n \geq 1} a_n \langle x, e_n \rangle \sigma_n \), is compact.

**Proof.** We consider the operators

\[
T_m(x) = \sum_{n=1}^{m} a_n \langle x, e_n \rangle \sigma_n, \quad m \in \mathbb{N}.
\]

By Bessel’s inequality \( \{T_m\}_{m \geq 1} \) is a sequence of bounded operators of finite rank

\[
\|T_m(x)\|^2 = \sum_{n=1}^{m} |a_n|^2 \|\langle x, e_n \rangle\|^2 \leq \|a\|_{\ell_{\infty}}^2 \|x\|^2.
\]
1.1. OPERATOR THEORY

For every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that $|a_n| < \epsilon$, $\forall n \geq N$.

$$\|(T - T_m)(x)\|^2 = \sum_{n>m} |a_n|^2 |\langle x, e_n \rangle|^2 \leq \epsilon^2 \|x\|^2, \forall m \geq N.$$ 

This implies that $\|T - T_m\| \to 0$. By the Proposition 1.1.1. the operator $T$ is compact.

**Proposition 1.1.7.** Let $T$ be a bounded operator on $H$. Then, $T$ is self-adjoint, i.e. $T = T^*$, if and only if $\langle T(x), x \rangle \in \mathbb{R}$, $\forall x \in H$.

**Proof.** If $T$ is self-adjoint and $x \in H$, then

$$\langle T(x), x \rangle = \langle x, T^*(x) \rangle = \langle x, T(x) \rangle = \overline{\langle T(x), x \rangle},$$

$$\langle T(x), x \rangle \in \mathbb{R}.$$ 

Conversely, we consider the operator $S = -i(T - T^*)$. $S$ is a bounded self-adjoint operator and

$$\langle S(x), x \rangle = -i \left( \langle T(x), x \rangle - \langle T^*(x), x \rangle \right) = -i \left( \langle T(x), x \rangle - \overline{\langle T(x), x \rangle} \right) = 0, \forall x \in H. \quad (1.3)$$

It is easy to prove the following polarization identity

$$\langle S(x), y \rangle = \frac{1}{4} \left( (\langle S(x + y), x + y \rangle - \langle S(x - y), x - y \rangle) \right. 
\left. + \frac{i}{4} (\langle S(x + iy), x + iy \rangle - \langle S(x - iy), x - iy \rangle) \right). \quad (1.4)$$

(1.3) and (1.4) imply that $\langle S(x), y \rangle = 0, \forall x, y \in H$. Follows that $S \equiv 0$ and as a consequence $T$ is self-adjoint.

**Proposition 1.1.8.** Let $T$ be a bounded self-adjoint operator on $H$, then

$$\|T\| = \sup_{\|x\| = 1} |\langle T(x), x \rangle|.$$ 

**Proof.** Without loss of generality we assume that $T \neq 0$. By the Cauchy-Schwarz inequality

$$|\langle T(x), x \rangle| \leq \|T\| \|x\|^2,$$

$$\|T\| \geq \sup_{\|x\| = 1} |\langle T(x), x \rangle| := C.$$ 

To prove the opposite inequality, let $x \in H$ such that $\|x\| = 1$, $T(x) \neq 0$ and let $y = \frac{T(x)}{\|T(x)\|}$. Then,

$$\|T(x)\| = \langle T(x), y \rangle = \frac{1}{4} \left( \langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle \right) \leq \frac{C}{4} (\|x + y\|^2 + \|x - y\|^2),$$

$$\|T\| = \sup_{\|x\| = 1} |\langle T(x), x \rangle| = \sup_{\|x\| = 1} \|x\|.$$ 


by the Parallelogram law
\[ \|T(x)\| \leq \frac{C}{4} \left( 2\|x\|^2 + 2\|y\|^2 \right) = C. \]

Follows that
\[ \|T\| = \sup_{\|x\|=1} |\langle T(x), x \rangle|. \]

\[ \square \]

**Definition 1.1.3.** An operator \( T \) on \( H \) called positive if \( \langle T(x), x \rangle \geq 0, \forall x \in H \).

**Lemma 1.1.1.** Let \( T \) be a compact self-adjoint operator on \( H \). Then

1. The eigenvalues of \( T \) are real and if \( T \) is positive, then the eigenvalues are non-negative.

2. If \( T \neq 0 \), then \( T \) has at least one non-zero eigenvalue, in fact \( \|T\| \) or \( -\|T\| \) is an eigenvalue.

3. The eigenvectors that correspond to distinct non-zero eigenvalues are orthogonal.

4. If \( T \) has infinitely many eigenvalues then it has countably many and the sequence of eigenvalues \( \{a_n\}_{n \geq 1} \) converges to 0.

5. The dimension of an eigenspace, that corresponds to a non-zero eigenvalue, is finite.

**Proof.**

1. Let \( \lambda \) be an eigenvalue of \( T \) and let \( x \in H \) be a corresponding unit eigenvector. Then,
\[ \lambda = \langle T(x), x \rangle \in \mathbb{R}. \]

Similarly, if \( T \) is positive, then \( \lambda \geq 0 \).

2. By the Proposition 1.1.8. there exists a sequence of unit vectors \( \{x_n\}_{n \geq 1} \subset H \) such that
\[ \lim_{n \to +\infty} \langle T(x_n), x_n \rangle = \|T\| \text{ or } \lim_{n \to +\infty} \langle T(x_n), x_n \rangle = -\|T\|. \]

Replacing \( T \) by \(-T\), if necessary, we can assume that
\[ \lim_{n \to +\infty} \langle T(x_n), x_n \rangle = \|T\|. \]
By the Cauchy-Schwarz inequality
\[
\langle T(x_n), x_n \rangle \leq \|T(x_n)\| \leq \|T\|,
\]
\[
\lim_{n \to +\infty} \|T(x_n)\| = \|T\|.
\]

We observe that
\[
\|T(x_n) - \|T\| x_n\|^2 = \|T(x_n)\|^2 + \|T\|^2 - 2 \|T\| \langle T(x_n), x_n \rangle,
\]
\[
\lim_{n \to +\infty} \|T(x_n) - \|T\| x_n\|^2 = 0.
\]

Since \(\{x_n\}_{n \geq 1}\) is bounded and \(T\) is compact there exists a subsequence \(\{x_{n_k}\}_{k \geq 1}\) and a vector \(y \in H\) such that
\[
\lim_{n \to +\infty} \|T(x_{n_k}) - y\| = 0.
\]

Follows that \(\lim_{n \to +\infty} x_{n_k} = \frac{y}{\|T\|}\) and \(T(y) = \|T\| y\).

3. Let \(\lambda_1, \lambda_2\) be distinct eigenvalues of \(T\) and let \(x_1, x_2 \in H\) be eigenvectors of \(\lambda_1, \lambda_2\) respectively
\[
\langle T(x_1), x_2 \rangle = \langle x_1, T(x_2) \rangle = \lambda_1 \langle x_1, x_2 \rangle = \lambda_2 \langle x_1, x_2 \rangle.
\]
Since \(\lambda_1 \neq \lambda_2\), we have that \(\langle x_1, x_2 \rangle = 0\).

4. Trivially 3. implies that the eigenvalues of \(T\) are countable many. We assume that there exists a \(\delta > 0\) and a subsequence \(\{a_{n_k}\}_{k \geq 1}\) such that
\[
|a_{n_k}| > \delta, \ \forall k \geq 1.
\]

By 3. we can choose an orthonormal sequence \(\{x_{n_k}\}_{k \geq 1}\) such that \(x_{n_k}\) is an eigenvector corresponding to \(a_{n_k}\). We will prove that \(\{x_{n_k}\}_{k \geq 1}\) converges weakly to 0. By the Proposition 1.1.4. there exists a weakly convergent further subsequence. It is sufficient to prove that 0 is the unique weak limit point. If \(\{x_{n_{k_l}}\}_{l \geq 1}\) converges weakly to \(x \in H\), then
\[
\|x\|^2 = \lim_{l \to +\infty} \langle x_{n_{k_l}}, x \rangle = \lim_{l \to +\infty} \lim_{m \to +\infty} \langle x_{n_{k_l}}, x_{n_{k_m}} \rangle = 0.
\]
Follows that \(\{x_{n_k}\}_{k \geq 1}\) converges weakly to 0 and by the Proposition 1.1.5.
\[
\delta^2 \leq \limsup_{k \to \infty} |a_{n_k}|^2 = \lim_{k \to \infty} \|T(x_{n_k})\|^2 = 0, \text{ contradiction.}
\]
So, \(\{a_n\}_{n \geq 1}\) converges to 0.
5. We assume that there exists a non-zero eigenvalue $\lambda$ with infinite dimensional corresponding eigenspace. Let $\{e_n\}_{n \geq 1}$ be an orthonormal basis of it. As above, $\{e_n\}_{n \geq 1}$ converges weakly to 0 and

$$|\lambda| = \lim_{n \to \infty} \|T(e_n)\| = 0,$$ contradiction.

\[ \square \]

**Theorem 1.1.2.** Let $T$ be a compact self-adjoint operator on $H$. Then, $T$ can be written in the form

$$T(x) = \sum_{n \geq 1} a_n \langle x, e_n \rangle e_n,$$

where $\{a_n\}_{n \geq 1}$ is the sequence of non-zero eigenvalues of $T$ counting the multiplicity and $\{e_n\}_{n \geq 1}$ is an orthonormal sequence of corresponding eigenvectors.

**Proof.** Let $\{b_1, b_2, \ldots\} \subset \mathbb{R}$ be the sequence of non-zero eigenvalues of $T$ such that

$$|b_1| > |b_2| > \ldots$$

and let $d_n < +\infty$ be the dimension of the eigenspace that corresponds to the eigenvalue $b_n$. We define the sequence $\{a_n\}_{n \geq 1}$ as

$$a_1 = \ldots = a_{d_1} = b_1$$

$$a_{d_1+1} = \ldots = a_{d_1+d_2} = b_2$$

$$a_{d_1+d_2+1} = \ldots = a_{d_1+d_2+d_3} = b_3$$

$$\vdots$$

Let $\{e_n\}_{n \geq 1}$ be the sequence that begins with an orthonormal basis for the eigenspace of $b_1$, followed by an orthonormal basis for the eigenspace of $b_2$, $\ldots$. By the Lemma 1.1.1. $\{e_n\}_{n \geq 1}$ is an orthonormal sequence and $T(e_n) = a_n e_n$. We consider the operator $T_0 : H \to H$

$$T_0(x) = \sum_{n \geq 1} a_n \langle x, e_n \rangle e_n, \ x \in H.$$

By the Proposition 1.1.6. $T_0$ is compact.

$$\langle T_0(x), y \rangle = \sum_{n \geq 1} a_n \langle x, e_n \rangle \langle e_n, y \rangle = \sum_{n \geq 1} a_n \langle y, e_n \rangle \langle e_n, x \rangle = \langle x, T_0(y) \rangle, \ x, y \in H.$$ So, $T_0$ is self-adjoint. Let $M = \text{span}\{e_1, e_2, \ldots\}$, it is easy to prove that $T(M) \subset M$ and $T(M^\perp) \subset M^\perp$. We define the operator $S := T - T_0$ and we observe that $S$ is
compact and self-adjoint. Let \( \lambda \) be an eigenvalue of \( S \) and let \( x \) be a corresponding non-zero eigenvector, then

\[
x - \sum_{n \geq 1} \langle x, e_n \rangle e_n \in M^\perp,
\]

\[
M^\perp \ni T(x - \sum_{n \geq 1} \langle x, e_n \rangle e_n) = T(x) - T_0(x) = S(x) = \lambda x.
\]

Thus, \( x \in M^\perp \) and \( T(x) = \lambda x \). \( \lambda \) is an eigenvalue of \( T \) with eigenvector \( x \not\in M \), this implies that \( \lambda = 0 \). We proved that \( S \) is a compact self-adjoint operator with no non-zero eigenvalues. By the Lemma 1.1.1. follows that \( S \equiv 0 \) and as a consequence \( T \equiv T_0 \),

\[
T(x) = \sum_{n \geq 1} a_n \langle x, e_n \rangle e_n.
\]

\[
\square
\]

**Definition 1.1.4.** For an operator \( T \) on \( H \), we say that it admits \( n \)th – root, \( n \in \mathbb{N} \) if there exists an operator \( R \) on \( H \) such that \( R^n = T \). We will denote such an operator by \( T^{\frac{1}{n}} \). For \( n = 2 \), the operator \( T^{\frac{1}{2}} \) (if exists) called the square root of \( T \).

**Proposition 1.1.9.** Let \( T \) be a compact operator on \( H \). Then, the operator \( T^*T \) is a positive compact operator, which admits a unique compact and positive \( m \)th – root, \( \forall m \in \mathbb{N} \).

**Proof.** It is easy to prove that \( T^*T \) is a positive compact operator. By the Lemma 1.1.1. the eigenvalues of \( T^*T \) are non-negative and by the Theorem 1.1.2. it can be written in the form

\[
T^*T(x) = \sum_{n \geq 1} a_n \langle x, e_n \rangle e_n,
\]

where \( \{a_n\}_{n \geq 1} \) is the sequence of positive eigenvalues of \( T^*T \). We define the operator \( S : H \to H \) as

\[
S(x) = \sum_{n \geq 1} a_n^m \langle x, e_n \rangle e_n.
\]

It easy to prove that \( S \) is compact, positive and \( S^{(m)} = T^*T \).

For uniqueness, we assume that there exists another such an operator \( F \). By the Theorem 1.2.2. \( F \) can be written in the form

\[
F(x) = \sum_{n \geq 1} b_n \langle x, \sigma_n \rangle \sigma_n,
\]

where \( \{b_n\}_{n \geq 1} \) is the sequence of positive eigenvalues of \( F \) counting the multiplicity and \( \{\sigma_n\}_{n \geq 1} \) is an orthonormal sequence of corresponding eigenvectors.

\[
F^{(m)}(\sigma_n) = b_n^m \sigma_n = T^*T(\sigma_n) = S^{(m)}(\sigma_n).
\]
We observe that $b_n$ is an eigenvalue of $S$ and $\sigma_n$ is a corresponding eigenvector. So, $F$, $S$ agree on the orthonormal sequence $\{\sigma_n\}_{n \geq 1}$ and

$$T^*T(x) = \sum_{n \geq 1} b_n^m \langle x, \sigma_n \rangle \sigma_n.$$ 

Follows that $\text{span}\{e_1, e_2, \ldots\} = \text{span}\{\sigma_1, \sigma_2, \ldots\}$ and $F \equiv S$. 

**Definition 1.1.5.** Let $T$ be a compact operator on $H$. We denote by $|T|$ the unique compact and positive square root of the operator $T^*T$. The non-increasing sequence $\{s_n(T)\}_{n \geq 1} = \{s_n\}_{n \geq 1}$ of the positive eigenvalues of $|T|$ called the sequence of singular values of $T$ and $s_n(T)$ called the $n$th singular value.

**Definition 1.1.6.** An operator $U$ on $H$ is called a partial isometry if it is an isometry on the orthogonal complement of its kernel.

**Theorem 1.1.3. (Polar Decomposition)** Let $T$ be a compact operator on $H$. Then, there exists a partial isometry $U : H \to H$ such that

$$T = U|T|.$$ 

Moreover, $\text{Ker}U = \text{Ker}T$.

**Proof.** As above, $T^*T$ and $|T|$ can be written as

$$T^*T(x) = \sum_{n \geq 1} s_n^2 \langle x, e_n \rangle e_n \quad \text{and} \quad |T|(x) = \sum_{n \geq 1} s_n \langle x, e_n \rangle e_n,$$

where $\{s_n\}_{n \geq 1}$ is the sequence of singular values of $T$ and $\{e_n\}_{n \geq 1}$ is the orthonormal sequence of the corresponding eigenvectors. We observe that

$$\|T(x)\|^2 = \langle T^*T(x), x \rangle = \langle |T|^{(2)}(x), x \rangle = \langle |T|(x), |T|(x) \rangle = \| |T|(x) \|^2$$

and $\text{Ker}T = \text{Ker}|T| = M^\perp$, where $M = \overline{\text{span}\{e_1, e_2, \ldots\}}$. The operator $|T|$ is injective on $M$ and $|T|(\text{span}\{e_1, e_2, \ldots\}) = \text{span}\{e_1, e_2, \ldots\}$. We define the operator $U$ as

$$U = T|T|^{-1}, \text{ on } \text{span}\{e_1, e_2, \ldots\} \quad \text{and} \quad U = 0, \text{ on } M^\perp.$$ 

For a vector $y \in \text{span}\{e_1, e_2, \ldots\}$, there exists a $x \in \text{span}\{e_1, e_2, \ldots\}$ such that $y = |T|(x)$ and

$$\|U(y)\| = \| T|T|^{-1}(y) \| = \| |T|^{-1}(y) \| = \|y\|.$$

By the Hahn-Banach theorem $U$ has an extension on $H$ that we will denote again by $U$. Follows that it is a unique partial isometry, since

$$M = M^{\perp \perp} = \text{Ker}T^\perp \text{ (M is closed)}.$$ 

Moreover, $U|T|(x) = T(x), \forall x \in M$ and $T(x) = 0 = U|T|(x), \forall x \in M^\perp$. So,

$$T = U|T|.$$

$\blacksquare$
Theorem 1.1.4. A compact operator \(T\) on \(H\) can be written in the form
\[
T(x) = \sum_{n \geq 1} s_n(x, e_n)\sigma_n,
\]
where \(\{s_n\}_{n \geq 1}\) is the sequence of singular values of \(T\) and \(\{e_n\}_{n \geq 1}, \{\sigma_n\}_{n \geq 1}\) are orthonormal sequences.

Proof. By the Theorem 1.1.2, there exists an orthonormal sequence \(\{e_n\}_{n \geq 1}\) such that
\[
|T|(x) = \sum_{n \geq 1} s_n(x, e_n)e_n
\]
and by the Theorem 1.1.3, there exists a unique partial isometry \(U\) on \(H\) such that
\[
T(x) = U|T|(x) = \sum_{n \geq 1} s_n(x, e_n)\sigma_n,
\]
where \(\sigma_n = U(e_n)\). It remains to prove that \(\{\sigma_n\}_{n \geq 1}\) is orthonormal. Trivially the vector \(\sigma_n\) is unit for every \(n \in \mathbb{N}\)
\[
\|\sigma_n\| = \|U(e_n)\| = \|e_n\| = 1.
\]

For \(n \neq m\),
\[
\langle \sigma_n, \sigma_m \rangle = \langle U(e_n), U(e_m) \rangle = \langle |T|^{-1}(e_n), T|T|^{-1}(e_m) \rangle
= \langle |T|^{-1}(e_n), T^*T|T|^{-1}(e_m) \rangle
= \langle |T|^{-1}(e_n), |T|(e_m) \rangle = \langle |T| |T|^{-1}(e_n), e_m \rangle = \langle e_n, e_m \rangle = 0.
\]

Definition 1.1.7. Let \(T\) be a compact operator and \(\{s_n\}_{n \geq 1}, \{e_n\}_{n \geq 1}, \{\sigma_n\}_{n \geq 1}\) as above, then \(T(x) = \sum_{n \geq 1} s_n(x, e_n)\sigma_n\) called the canonical decomposition of \(T\).

Proposition 1.1.10. Let \(T\) be a compact operator. Then, \(T\) and \(T^*\) has the same sequence of singular values and if \(T(x) = \sum_{n \geq 1} s_n(x, e_n)\sigma_n\) is the canonical decomposition of \(T\), then \(T^*(x) = \sum_{n \geq 1} s_n(x, \sigma_n)e_n\) is the canonical decomposition of \(T^*\).

Proof. Let \(\lambda\) be a positive eigenvalue of \(T^*T\) and \(x \in H\) be a corresponding unit eigenvector, then
\[
T^*T(x) = \lambda x \quad \text{and} \quad TT^*(T(x)) = \lambda T(x).
\]
This implies that \(T\) and \(T^*\) has the same sequence of singular values. Let \(x, y \in H\),
\[
\langle T(x), y \rangle = \sum_{n \geq 1} s_n(x, e_n)\langle \sigma_n, y \rangle = \langle x, \sum_{n \geq 1} s_n(y, \sigma_n)e_n \rangle,
\]
follows that \(T^*(x) = \sum_{n \geq 1} s_n(x, \sigma_n)e_n\) is the canonical decomposition of \(T^*\).
1.1.2 Schatten Classes

Definition 1.1.8. For $0 < p < +\infty$, the $p$–Schatten class of operators on $H$ is defined by

$$S_p = \{T \text{ compact} : s(T) = (s_n(T))_{n \geq 1} \text{ belongs to the } \ell^p \text{ space} \}.$$  

Suppose $p \in [1, +\infty)$, for an operator $T \in S_p$ with sequence of singular values $s(T) = (s_n(T))_{n \geq 1}$, the $p$–Schatten norm of it (we will prove later that it is a norm) is defined by

$$\|T\|_{S_p} = \|s(T)\|_{\ell^p} = \left(\sum_{n \geq 1} |s_n|^p\right)^{\frac{1}{p}}.$$  

We will denote by $S_\infty$ the space of all bounded operators on $H$ equipped with the operator-norm. The class $S_2$ called the Hilbert-Schmidt class and the class $S_1$ called the trace class.

Theorem 1.1.5. Let $T$ be a compact operator on $H$. Then,

$$s_{n+1}(T) = \inf\{ \|T - F\| : F \text{ is an operator on } H \text{ with } \operatorname{rank}(F) \leq n \}. $$

Proof. The canonical decomposition of the operator is $T(x) = \sum_{k \geq 1} s_k \langle x, e_k \rangle \sigma_k$.

We define the operator $T_n(x) = \sum_{k=1}^n s_k \langle x, e_k \rangle \sigma_k$ and by Bessel’s inequality

$$\|T - T_n\| = \left\| \sum_{k \geq n+1} s_k \langle x, e_k \rangle \sigma_k \right\|^2 = \sum_{k \geq n+1} |s_k |^2 \leq s_{n+1}^2 \|x\|^2,$$

$$(T - T_n)(e_{n+1}) = s_{n+1}.$$ 

Follows that

$$s_{n+1} = \|T - T_n\| \geq \inf\{ \|T - F\| : F \text{ is an operator on } H \text{ with } \operatorname{rank}(F) \leq n \}. \hspace{1cm} (1.5)$$

Let $F$ be an arbitrary operator on $H$ with $\operatorname{rank}(F) \leq n$. The dimension of the image $F(\overline{\text{span}\{e_1, \ldots, e_{n+1}\}})$ is less or equal to $n$, so there exists a unit vector $x_0 \in \overline{\text{span}\{e_1, \ldots, e_{n+1}\}}$ such that $F(x_0) = 0$.

$$\|T - F\|^2 \geq \|(T - F)(x_0)\|^2 = \left\| T \left( \sum_{k=1}^{n+1} \langle x_0, e_k \rangle e_k \right) \right\|^2 = \sum_{k=1}^{n+1} s_k^2 |\langle x_0, e_k \rangle|^2 \geq s_{n+1}^2,$$

$$s_{n+1} \leq \inf\{ \|T - F\| : F \text{ is an operator on } H \text{ with } \operatorname{rank}(F) \leq n \}. \hspace{1cm} (1.6)$$

The proof follows by (1.5) and (1.6). ■
Proposition 1.1.11. Suppose $T$, $S$, $L$ are bounded operators on $H$ and $n$, $m$ are non-negative integers.

1. If $T$, $S$ are compact, then

$$s_{n+m+1}(T + S) \leq s_{n+1}(T) + s_{m+1}(S).$$

2. If $T$, $S$ are compact, then

$$s_{n+m+1}(TS) \leq s_{n+1}(T)s_{m+1}(S).$$

3. If $T$ is compact, then

$$s_{n+1}(LTS) \leq \|L\|s_{n+1}(T)\|S\|.$$

Proof. We assume that $T$, $S$ are compact. Let $F_1$, $F_2$ be operators with $\text{rank}(F_1) \leq n$, $\text{rank}(F_2) \leq m$. Then,

$$s_{n+m+1}(T + S) \leq \|T + S - F_1 - F_2\| \leq \|T - F_1\| + \|S - F_2\|,$$

$$s_{n+m+1}(TS) \leq \|TS - TF_2 - F_1(S - F_2)\| = \|(T - F_1)(S - F_2)\| \leq \|T - F_1\|\|S - F_2\|.$$

By the Theorem 1.1.5.

$$s_{n+m+1}(T + S) \leq s_{n+1}(T) + s_{m+1}(S),$$

$$s_{n+m+1}(TS) \leq s_{n+1}(T)s_{m+1}(S).$$

We assume now that $T$ is compact. Let $F$ be an arbitrary operator with $\text{rank}(F) \leq n$. Then,

$$s_{n+1}(LTS) \leq \|LTS - LFS\| = \|L(T - F)S\| \leq \|L\|\|T - F\|\|S\|.$$

By the Theorem 1.1.5.

$$s_{n+1}(LTS) \leq \|L\|s_{n+1}(T)\|S\|.$$

Corollary 1.1.1. $S_p$ is a vector space for $p > 0$.

Corollary 1.1.2. If $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $T \in S_p$, $S \in S_q$, then $TS \in S_1$.

Proposition 1.1.12. Let $p > 0$ and $T$ be a compact operator on $H$. Then, $T \in S_p$ if and only if $|T| \in S_p$. Moreover,

$$\|T\|^p_{S_p} = \|T^*\|^p_{S_p} = \|T\|^p_{S_p} = \|T\|^p_{S_1} = \|T^*T\|^p_{S_p/2}.$$
The canonical decompositions of $|T|$, $T$, $T^*$, $T^* T$, $|T|^p$ are
\[
|T|(x) = \sum_{n \geq 1} s_n \langle x, e_n \rangle e_n, \quad T(x) = \sum_{n \geq 1} s_n \langle x, e_n \rangle \sigma_n, \quad T^*(x) = \sum_{n \geq 1} s_n \langle x, \sigma_n \rangle e_n, \quad T^* T(x) = \sum_{n \geq 1} s^2_n \langle x, e_n \rangle e_n, \quad |T|^p(x) = \sum_{n \geq 1} s^p_n \langle x, e_n \rangle e_n.
\]
The claim follows. ■

**Proposition 1.1.13. (Self-adjoint decomposition)** A compact operator $T$ on $H$ can be written in the form

$$T = T_1 + iT_2,$$

where $T_1, T_2$ are self-adjoint compact operators and for $p > 0$, $T \in S_p$ if and only if $T_1, T_2 \in S_p$.

**Proof.** The operators $T_1 = \frac{T + T^*}{2}$, $T_2 = \frac{T - T^*}{2i}$ are self-adjoint, compact and $T = T_1 + iT_2$. By the Proposition 1.1.11. and the Proposition 1.1.12. for $p > 0$, $T \in S_p$ if and only if $T_1, T_2 \in S_p$. ■

**Corollary 1.1.3.** Convergence in $S_p$ implies convergence in $S_\infty$.

**Proof.** Let $\{T_n\}_{n \geq 1}$ be a sequence that converges to $T$ in $S_p$, we observe that

$$T_{1,n} := \frac{T_n + T_n^*}{2} \to T_1 := \frac{T + T^*}{2} \quad \text{and} \quad T_{2,n} := \frac{T_n - T_n^*}{2i} \to T_2 := \frac{T - T^*}{2i} \quad \text{in } S_p.$$

By the Lemma 1.1.1. and the Proposition 1.1.11. the norm of a self-adjoint operator is the first singular value. So,

$$T_{1,n} \to T_1 \quad \text{and} \quad T_{2,n} \to T_2 \quad \text{in } S_\infty.$$

This completes the proof. ■

**Proposition 1.1.14. (Positive decomposition)** A compact operator $T$ on $H$ can be written in the form

$$T = P_1 - P_2 + i(P_3 - P_4),$$

where $P_1, P_2, P_3, P_4$ are positive compact operators and for $p > 0$, $T \in S_p$ if and only if $P_1, P_2, P_3, P_4 \in S_p$.

**Proof.** By the Proposition 1.1.13. it is sufficient to prove that every self-adjoint operator $S$ can be written in the form $S = S_1 - S_2$, where $S_1, S_2$ are positive compact operators and for $p > 0$, $S \in S_p$ if and only if $S_1, S_2 \in S_p$.

We observe that $S_1 = \frac{S + |S|}{2}$, $S_2 = \frac{|S|-S}{2}$ are positive, compact operators and $S = S_1 - S_2$. By the Proposition 1.1.11. and the Proposition 1.1.12. for $p > 0$, $S \in S_p$ if and only if $S_1, S_2 \in S_p$. ■
1.1. OPERATOR THEORY

**Theorem 1.1.6.** Let $T$ be a compact operator on $H$ and $\{x_n\}_{n \geq 1}$ be an orthonormal basis of $H$. Then,

$$\|T\|_{S_2}^2 = \sum_{n \geq 1} \|T(x_n)\|^2.$$

**Proof.** The canonical decomposition of the operator is

$$T(x) = \sum_{n \geq 1} s_n \langle x, e_n \rangle \sigma_n.$$

By Tonelli’s theorem for counting measure and Parseval’s formula

$$\sum_{n \geq 1} \|T(x_n)\|^2 = \sum_{n \geq 1} \sum_{m \geq 1} s_m^2 |\langle x_n, e_m \rangle|^2 = \sum_{m \geq 1} \sum_{n \geq 1} s_m^2 |\langle x_n, e_m \rangle|^2 = \sum_{m \geq 1} s_m^2 \|e_m\|^2 = \|T\|_{S_2}^2.$$  

■

**Theorem 1.1.7.** Let $T$ be a positive compact operator on $H$ and $\{x_n\}_{n \geq 1}$ be an orthonormal basis of $H$. Then,

$$\|T\|_{S_1} = \sum_{n \geq 1} \langle T(x_n), x_n \rangle.$$

**Proof.** The canonical decomposition of the operator is

$$T(x) = \sum_{n \geq 1} s_n \langle x, e_n \rangle e_n.$$

By Tonelli’s theorem for counting measure and Parseval’s formula

$$\sum_{n \geq 1} \langle T(x_n), x_n \rangle = \sum_{n \geq 1} \sum_{m \geq 1} s_m |\langle x_n, e_m \rangle|^2 = \sum_{m \geq 1} \sum_{n \geq 1} s_m^2 |\langle x_n, e_m \rangle|^2 = \sum_{m \geq 1} s_m \|e_m\|^2 = \|T\|_{S_1}.$$

■

**Proposition 1.1.15.** Let $T$ be a compact operator on $H$ that belongs to the trace class $S_1$ and let $\{x_n\}_{n \geq 1}$ be an orthonormal basis of $H$. Then, the sum $\sum_{n \geq 1} \langle T(x_n), x_n \rangle$ converges absolutely and it is independent of the choice of orthonormal basis.

**Proof.** By the Proposition 1.1.14. we can assume that $T$ is positive, the proof follows from the Theorem 1.1.7. ■

**Definition 1.1.9.** For an operator $T$ that belongs to the trace class $S_1$, the trace is defined by

$$tr(T) = \sum_{n \geq 1} \langle T(x_n), x_n \rangle,$$

where $\{x_n\}_{n \geq 1}$ is an orthonormal basis of $H$. 
Proposition 1.1.16. Let $T \in S_1$, then $|tr(T)| \leq \|T\|_{S_1}$.

Proof. Suppose the canonical decomposition

$$T(x) = \sum_{n \geq 1} s_n \langle x, e_n \rangle \sigma_n.$$ 

Let $\{x_n\}_{n \geq 1}$ be an orthonormal basis. Then,

$$|tr(T)| = \left| \sum_{m \geq 1} \langle T(x_m), x_m \rangle \right| \leq \sum_{m \geq 1} \sum_{n \geq 1} s_n |\langle x_m, e_n \rangle| |\langle \sigma_n, x_m \rangle| \leq \sum_{n \geq 1} s_n \|\sigma_n\| \|e_n\| = \|T\|_{S_1}.$$ 

Proposition 1.1.17. Let $T \in S_p$, $S \in S_q$, where $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$tr(TS) = tr(ST).$$ 

Proof. By the Corollary 1.1.2. $TS, ST \in S_1$. Let $\{x_n\}_{n \geq 1}$ be an orthonormal basis, then

$$tr(ST) = \sum_{n \geq 1} \langle ST(x_n), x_n \rangle = \sum_{n \geq 1} \langle T(x_n), S^*(x_n) \rangle$$

$$= \sum_{n \geq 1} \sum_{m \geq 1} \langle T(x_n), x_m \rangle \langle S^*(x_n), x_m \rangle$$

$$= \sum_{m \geq 1} \sum_{n \geq 1} \langle S(x_m), x_n \rangle \langle T^*(x_m), x_n \rangle$$

$$= tr(TS).$$ 

Lemma 1.1.2. Let $T$ be a positive operator on $H$, then

$$|\langle T(x), y \rangle| \leq \langle T(x), x \rangle + \langle T(y), y \rangle, \forall x, y \in H.$$ 

Proof. For $x, y \in H$

$$0 \leq \langle T(x - y), x - y \rangle = \langle T(x), x \rangle + \langle T(y), y \rangle - 2Re(\langle T(x), y \rangle).$$ 

Replacing $y$ with $-y$ and then $y$ with $iy$

$$|Re(\langle T(x), y \rangle)| \leq \frac{1}{2} (\langle T(x), x \rangle + \langle T(y), y \rangle),$$

$$|Im(\langle T(x), y \rangle)| \leq \frac{1}{2} (\langle T(x), x \rangle + \langle T(y), y \rangle).$$

Follows that

$$|\langle T(x), y \rangle| \leq \langle T(x), x \rangle + \langle T(y), y \rangle.$$
1.1. OPERATOR THEORY

**Theorem 1.1.8.** Let $T$ be a compact operator on $H$ and $p \geq 1$. Then, $T \in S_p$ if and only if $\sum_{n \geq 1} |\langle T(x_n), y_n \rangle|^p < \infty$, for every orthonormal sequences $\{x_n\}_{n \geq 1}$, $\{y_n\}_{n \geq 1}$.

**Proof.** By the Proposition 1.1.14, we can assume that $T$ is a positive compact operator with canonical decomposition

$$ T(x) = \sum_{n \geq 1} s_n \langle x, e_n \rangle e_n. $$

Suppose $T \in S_p$ and $\{x_n\}_{n \geq 1}$, $\{y_n\}_{n \geq 1}$ are orthonormal. For an arbitrary unit vector $x \in H$,

$$ \langle T(x), x \rangle = \sum_{n \geq 1} s_n |\langle x, e_n \rangle|^2 \leq \left( \sum_{n \geq 1} s_n^p |\langle x, e_n \rangle|^2 \right)^{\frac{1}{p}} \left( \sum_{n \geq 1} |\langle x, e_n \rangle|^2 \right)^{\frac{1}{q}}, $$

$$ \langle T(x), x \rangle \leq \left( \sum_{n \geq 1} s_n^p |\langle x, e_n \rangle|^2 \right)^{\frac{1}{p}}, \quad \frac{1}{p} + \frac{1}{q} = 1. $$

By Tonelli's theorem for counting measure and Bessel's inequality

$$ \sum_{n \geq 1} |\langle T(x_n), x_n \rangle|^p \leq \sum_{n \geq 1} \sum_{m \geq 1} s_m^p |\langle x_n, e_m \rangle|^2 \leq \sum_{m \geq 1} s_m^p \|e_m\|^2 = \sum_{m \geq 1} s_m^p = \|T\|_{S_p}^p. \quad (1.7) $$

By (1.7) and the Lemma 1.1.2.

$$ \sum_{n \geq 1} |\langle T(x_n), y_n \rangle|^p \leq 2^p \left( \sum_{n \geq 1} |\langle T(x_n), x_n \rangle|^p + \sum_{n \geq 1} |\langle T(y_n), y_n \rangle|^p \right) \leq 2^{p+1} \|T\|_{S_p}^p < +\infty. $$

Conversely, for $\{x_n\}_{n \geq 1} = \{y_n\}_{n \geq 1} = \{e_n\}_{n \geq 1}$

$$ \|T\|_{S_p}^p = \sum_{n \geq 1} s_n^p = \sum_{n \geq 1} |\langle T(e_n), e_n \rangle|^p < +\infty. $$

\(\blacksquare\)

**Theorem 1.1.9.** Let $T$ be a compact operator on $H$ and $p \geq 2$. Then, $T \in S_p$ if and only if $\sum_{n \geq 1} \|T(x_n)\|^p < +\infty$, for every orthonormal sequence $\{x_n\}_{n \geq 1}$.

**Proof.** We observe that

$$ \sum_{n \geq 1} \|T(x_n)\|^p = \sum_{n \geq 1} \left| \langle T^* T(x_n), x_n \rangle \right|^p. $$

By the Theorem 1.1.8.

$$ \sum_{n \geq 1} \|T(x_n)\|^p < +\infty, \quad \text{for every orthonormal sequence } \{x_n\}_{n \geq 1}, \text{ if and only if} $$

$$ \sum_{n \geq 1} \left| \langle T^* T(x_n), x_n \rangle \right|^{\frac{p}{2}} < +\infty, \quad \text{for every orthonormal sequence } \{x_n\}_{n \geq 1}, \text{ if and only if} $$

$T^* T \in S_{\frac{p}{2}}$ if and only if $T \in S_p$, $p \geq 2$. \(\blacksquare\)
Lemma 1.1.3. (Horn) Let $T$, $S$ be compact operators on $H$. Then, for every $n \in \mathbb{N}$

$$\prod_{k=1}^{n} s_k(ST) \leq \prod_{k=1}^{n} s_k(T)s_k(S).$$

Proof. Working as in the proof of the existence of the polar decomposition we can prove that there exists a partial isometry $U$ on $H$ such that $|ST| = UST$. Let $K = \text{span}\{e_1, e_2, \ldots, e_n\}$ where $e_i$ is a unit eigenvector corresponding to the singular value $s_i(ST)$, $i = 1, \ldots, n$ such that $\dim(K) = n$ and let

$$I_K : K \to H, \quad I_K(x) = x, \quad x \in K.$$ 

$$I_{T(K)} : T(K) \to H, \quad I_{T(K)}(x) = x, \quad x \in T(K).$$ 

$$P_K : H = K \oplus K^\perp \to K, \quad P_K(x \oplus x^\perp) = x, \quad x \in K, \quad x^\perp \in K^\perp.$$ 

$$P_{T(K)} : H = T(K) \oplus T(K)^\perp \to T(K), \quad P_{T(K)}(x \oplus x^\perp) = x, \quad x \in T(K), \quad x^\perp \in T(K)^\perp.$$ 

Then,

$$\prod_{k=1}^{n} s_k(ST) = \det(P_K|ST|I_K)$$

$$= \det(P_KUSI_{T(K)}P_{T(K)}TI_K)$$

$$= \det(P_KSI_{T(K)}) \det(P_{T(K)}TI_K).$$

We consider the corresponding polar decompositions

$$P_KSI_{T(K)} = U_1|P_KSI_{T(K)}|, \quad P_{T(K)}TI_K = U_2|P_{T(K)}TI_K|.$$ 

$$\prod_{k=1}^{n} s_k(ST) = \det(U_1|P_KSI_{T(K)}|) \det(U_2|P_{T(K)}TI_K|)$$

$$\leq \det(|P_KSI_{T(K)}|) \det(|P_{T(K)}TI_K|)$$

$$= \prod_{k=1}^{n} s_k(P_KSI_{T(K)})s_k(P_{T(K)}TI_K)$$

$$\leq \prod_{k=1}^{n} s_k(T)s_k(S).$$

Lemma 1.1.4. (Hardy-Littlewood-Polya) Let $\phi : \mathbb{R} \to \mathbb{R}$ be an increasing convex function and $(a_n)_{n \geq 1}$, $(b_n)_{n \geq 1}$ be non-increasing real sequences such that

$$\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k, \quad \forall n \in \mathbb{N}.$$ 

Then,

$$\sum_{k=1}^{n} \phi(a_k) \leq \sum_{k=1}^{n} \phi(b_k), \quad \forall n \in \mathbb{N}.$$ 

Proof. [11]
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Theorem 1.1.10. Let $T \in S_p$, $S \in S_q$, where $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\|ST\|_{S_1} \leq \|T\|_{S_p} \|S\|_{S_q}.$$  

Proof. For $p = 1$ the proof follows by the Proposition 1.1.11. We assume now that $p, q \in (1, +\infty)$, by the Lemma 1.1.3. and the Lemma 1.1.4. for $\phi(x) = e^x$

$$\sum_{k=1}^{n} \log(s_k(ST)) \leq \sum_{k=1}^{n} \log(s_k(T)s_k(S)), \quad n \in \mathbb{N},$$

$$\sum_{k=1}^{n} s_k(ST) \leq \sum_{k=1}^{n} s_k(T)s_k(S), \quad n \in \mathbb{N}.$$  

The proof follows by Hölder's inequality. 

Corollary 1.1.4. Let $T \in S_p$, $S \in S_q$, where $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$|\text{tr}(ST)| \leq \|T\|_{S_p} \|S\|_{S_q}.$$  

Theorem 1.1.11. (Fan) Let $T, S$ be compact operators on $H$. Then, for every $n \in \mathbb{N}$

$$\sum_{k=1}^{n} s_k(T + S) \leq \sum_{k=1}^{n} (s_k(T) + s_k(S)).$$  

Proof. Suppose the polar decomposition

$$(S + T)(x) = U|S + T|(x) = \sum_{n \geq 1} s_n(S + T)\langle x, e_n \rangle U(e_n),$$

where $U$ is an isometric on $M := \text{span}\{e_1, e_2, \ldots, e_n\}$. We define the operators

$I_M : M \rightarrow H$, \quad $I_M(x) = x, \quad x \in M$ and

$P_M : H = M \oplus M^\perp \rightarrow M$, \quad $P_M(x \oplus x^\perp) = x, \quad x \in M, \quad x^\perp \in M^\perp$.

By the Corollary 1.1.4. and the Proposition 1.1.11.

$$\sum_{k=1}^{n} s_k(T + S) = \sum_{k=1}^{n} \langle (S + T)(e_k), U(e_k) \rangle = \sum_{k=1}^{n} \langle U^*(S + T)(e_k), e_k \rangle$$

$$= \sum_{k=1}^{n} \langle P_M U^*(S + T) I_M(e_k), e_k \rangle = \text{tr}(P_M U^* TI_M) + \text{tr}(P_M U^* SI_M)$$

$$\leq \|P_M U^* TI_M\|_{S_1} + \|P_M U^* SI_M\|_{S_1}$$

$$= \sum_{k=1}^{n} (s_k(P_M U^* TI_M) + s_k(P_M U^* SI_M))$$

$$\leq \sum_{k=1}^{n} (s_k(T) + s_k(S)).$$

□
Proposition 1.1.18. For $p \geq 1$, $\| \cdot \|_{S_p}$ is a norm on $S_p$.

**Proof.** The only property of the norm that it is not trivial to prove is the triangle inequality. Suppose $T, S \in S_p$, by the Theorem 1.1.11. and the Lemma 1.1.4. for $\phi(x) = x^p$

$$\sum_{n \geq 1} |s_n(T+S)|^p \leq \sum_{n \geq 1} |s_n(T) + s_n(S)|^p.$$ 

By Minkowski’s inequality

$$\| T + S \|_{S_p} \leq \| T \|_{S_p} + \| S \|_{S_p}.$$ 

\[\square\]

Lemma 1.1.5. Let $p \in [1, +\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $T$ be a bounded operator on $H$. Then, $T \in S_p$ if and only if $\sup \{|tr(S^*T)| : \|S\|_{S_q} = 1\} < +\infty$ and in this case

$$\| T \|_{S_p} = \sup \{|tr(S^*T)| : \|S\|_{S_q} = 1\} = \sup \{|tr(S^*T)| : \|S\|_{S_q} = 1, \text{rank}(S) < +\infty\}.$$ 

**Proof.** For $p \in [1, +\infty)$ the condition $\sup \{|tr(S^*T)| : \|S\|_{S_q} = 1\} < +\infty$ implies compactness [26]. So, in this case we can assume that $T$ is compact with canonical decomposition

$$T(x) = \sum_{n \geq 1} s_n(x, e_n)\sigma_n.$$ 

By the Corollary 1.1.4.

$$\| T \|_{S_p} \geq \sup \{|tr(S^*T)| : \|S\|_{S_q} = 1\}. \quad (1.8)$$

If $p \in (1, +\infty)$, we consider the finite rank operator

$$T_m^* = \sum_{k=1}^m \frac{p}{q} \frac{s_k^p(x, \sigma_k)e_n}{\left(\sum_{k=1}^m s_k^p\right)^{\frac{1}{q}}}.$$ 

We observe that

$$\| T_m^* \|_{S_q} = \| T_m \|_{S_q} = 1 \quad \text{and} \quad tr(T_m^*T) = \left(\sum_{k=1}^m s_k^p\right)^{\frac{1}{p}}.$$ 

$$\| T \|_{S_p} \leq \sup \{|tr(S^*T)| : \|S\|_{S_q} = 1 \text{ and } \text{rank}(S) < +\infty\}, \quad p \in (1, +\infty). \quad (1.9)$$
1.1. OPERATOR THEORY

If $p = 1$, we consider the operator

$$F_m^*(x) = \sum_{k=1}^{m} \langle x, \sigma_k \rangle e_k.$$  

$$\|F_m^*\| = \|F_m\| = 1 \quad \text{and} \quad tr(F_m^* T) = \sum_{k=1}^{m} s_k.$$  

$$\|T\|_{S_p} \leq \sup \{|tr(S^* T)| : \|S\|_{S_q} = 1 \text{ and } rank(S) < +\infty\}, \quad p = 1. \quad (1.10)$$

If $p = +\infty$, we observe that

$$\|T\| = \sup \{|\langle T(x), y \rangle| : \|x\| = \|y\| = 1\}.$$  

For arbitraries unit vectors $x, y$, we consider the operator

$$T_{x,y}^*(z) = \langle z, y \rangle x.$$  

We observe that

$$\|T_{x,y}^*\| = \|T_{x,y}\| = 1 \quad \text{and} \quad |tr(T_{x,y}^* T)| = |\langle T(x), y \rangle|.$$  

$$\|T\|_{S_p} \leq \sup \{|tr(S^* T)| : \|S\|_{S_q} = 1 \text{ and } rank(S) < +\infty\}, \quad p = +\infty. \quad (1.11)$$

The proof follows from (1.8), (1.9), (1.10) and (1.11).  

\[\Box\]

**Theorem 1.1.12.** $(S_p, \|\cdot\|_{S_p})$ is a Banach space for $p \geq 1$ and it is separable if $p \in [1, +\infty)$.

**Proof.** For $p = +\infty$ the proof is trivial. Suppose $p \in [1, +\infty)$ and $(T_n)_{n \geq 1}$ is a Cauchy sequence in $S_p$. For every $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that for every finite rank operator $S$ with norm $\|S\|_{S_q} = 1$, $\frac{1}{p} + \frac{1}{q} = 1$

$$|tr \left( S^* (T_n - T_m) \right)| \leq \epsilon, \quad n, m \geq n_0.$$  

By the Corollary 1.1.3. the sequence $(T_n)_{n \geq 1}$ converges to an operator $T \in S_\infty$. Letting $m \to +\infty$ in the equation above, we obtain that $T \in S_p$. Let $S$ be a finite rank operator with norm $\|S\|_{S_q} = 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$|tr \left( S^* (T_n - T) \right)| \leq \epsilon, \quad n \geq n_0.$$  

By the Lemma 1.1.5. follows that $T_n \to T$ in $S_p$ and as a consequence $(S_p, \|\cdot\|_{S_p})$ is a Banach space. Separability follows, working as in the case of $\ell^p$ spaces, from the fact that finite rank operators are dense in $S_p$, $p \in [1, +\infty)$,

$$T_m := \sum_{n=1}^{m} s_n(T) \langle x_n, e_n \rangle \sigma_n \to T, \quad \text{in } S_p.$$  

\[\Box\]
Corollary 1.1.5. The space $S_2$ is a separable Hilbert space equipped with the following inner product
\[
(T, S)_{S_2} = \text{tr}(S^* T), \quad T, S \in S_2.
\]

Theorem 1.1.13. Let $p \in [1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $S_p^* = S_q$ and the duality is given by the pairing
\[
(T, S) = \text{tr}(S^* T).
\]

Proof. We consider the operator $\phi : S_q \to S_p^*$, defined as
\[
\phi(S) = \langle \cdot, S \rangle = \text{tr}(S^* \cdot).
\]

By the Lemma 1.1.5. we obtain that $\phi$ is an isometry. It is sufficient to prove that $\phi$ is surjective. Let $L \in S_p^*$, by the Riesz representation theorem every rank-one operator on $H$ has the following form
\[
T_{x,y}(z) = \langle z, y \rangle x.
\]

We define $g : H \times H \to \mathbb{C}$ as $g(x, y) = L(T_{x,y})$ and we observe that
\[
|g(x, y)| \leq \|L\| \|T_{x,y}\|_{S_p} = \|L\| \|y\| \|x\|.
\]

This implies that $g(\cdot, y)$ and $\overline{g(x, \cdot)}$ are bounded linear functionals. By the Riesz representation theorem there exists a linear operator $F$ on $H$ such that
\[
g(x, y) = L(T_{x,y}) = \langle x, F(y) \rangle.
\]

\[
\|F\| = \sup \{|\langle x, F(y) \rangle| : \|x\| = \|y\| = 1\} \leq \|L\| \text{ and } L(T_{x,y}) = \text{tr}(F^* T_{x,y}).
\]

By linearity of the trace and the Lemma 1.1.5.
\[
L(T) = \text{tr}(F^* T), \quad \text{for every finite rank operator } T \in S_p.
\]

\[
+ \infty > \|L\| \geq \sup \left\{|\text{tr}(T^* F)| : \|T\|_{S_p} = 1 \text{ and } \text{rank}(T) < +\infty\right\} = \|F\|_{S_q}.
\]

By the continuity of the trace
\[
\phi(F) = L.
\]
Lemma 1.1.6. (The Marcinkiewicz Interpolation Theorem) Let \((X, \mu), (Y, \nu)\) be \(\sigma\)-finite measure spaces and \(T\) be a linear operator from \(L^1(X, \mu) + L^\infty(X, \mu)\) to \(\nu\)-measurable functions, such that

1. \(T\) maps \(L^\infty(X, \mu)\) boundedly into \(L^\infty(Y, \nu)\).

2. There exists a constant \(C > 0\) such that
\[
\nu(\{y : |T(f)(y)| > t\}) \leq \frac{C \|f\|_{L^1}^p}{t}, \quad \forall t > 0, \ \forall f \in L^1(X, \mu).
\]

Then, \(T\) maps \(L^p(X, \mu)\) boundedly into \(L^p(Y, \nu)\), for every \(p \in (1, +\infty)\).

**Proof.** Without loss of generality we can assume that
\[
\|T(f)\|_{L^\infty} \leq \|f\|_{L^s}, \quad \forall f \in L^\infty(X, \mu).
\]

For a function \(f \in L^p(x, \mu), p > 1\) and \(t > 0\), we consider the functions

\[
f_0(x) = f(x)\chi_{\{|w|:f(x)| > \frac{1}{2}\}}(x) \quad \text{and} \quad f_1(x) = f(x)\chi_{\{|w|:f(x)| \leq \frac{1}{2}\}}(x).
\]

We observe that
\[
\{y \in Y : |T(f_0(y))| > t\} \subseteq \left\{y \in Y : |T(f_1(y))| > \frac{t}{2}\right\} \cup \left\{y \in Y : |T(f_1(y))| > \frac{t}{2}\right\}
\]

and as a consequence
\[
\nu(\{y : |T(f(y))| > t\}) \leq \nu\left(\left\{y : |T(f_0(y))| > \frac{t}{2}\right\}\right) \leq \frac{2C \|f_0\|_{L^1}^p}{t}.
\]

By Fubini’s theorem and (1.12)
\[
\int_Y |T(f)|^p d\nu(y) = \int_Y \int_0^{\|T(f(y))\|} pt^{p-1} dt d\nu(y) = \int_Y \int_0^{+\infty} pt^{p-1}\chi_{\{|T(f(y))| > t\}} dt d\nu(y)
\]

\[
= \int_0^{+\infty} pt^{p-1}\nu(\{y : |T(f(y))| > t\}) dt \leq \int_0^{+\infty} pt^{p-1} \frac{2C \|f_0\|_{L^1}^p}{t} dt
\]

\[
= 2C \int_X |f(x)| \int_0^{2|f(x)|} t^{p-2} dt d\mu(x) = C \frac{p}{p-1} 2^p \|f\|_{L^p}^p.
\]

So, \(T\) maps \(L^p(X, \mu)\) boundedly into \(L^p(Y, \nu)\) for every \(p \in (1, +\infty)\).
Theorem 1.1.14. Let \((X, \mu)\) be a \(\sigma\)-finite measure space and \(\phi : S_\infty \to L^\infty(X, \mu)\) be a bounded operator. If there exists a constant \(C > 0\), such that
\[
\mu\left(\{x : \phi(T)(x) > t\}\right) \leq \frac{C}{t} \|T\|_{S_1}, \quad \forall t > 0, \quad \forall T \in S_1.
\]
Then, \(\phi\) maps \(S_p\) boundedly into \(L^p(X, \mu)\), for every \(p \in (1, +\infty)\).

Proof. Suppose \(T \in S_p\), \(p \in (1, +\infty)\) with canonical decomposition
\[
T(x) = \sum_{n \geq 1} s_n(x, e_n)\sigma_n.
\]
We consider the operator \(\psi\) from \(\ell^p\) into the space of \(\mu\)-measurable functions
\[
\psi([a_n]_{n \geq 1}) = \phi\left(\sum_{n \geq 1} a_n(\cdot, e_n)\sigma_n\right).
\]
\(\phi\) maps \(\ell_\infty\) boundedly into \(L^\infty(X, \mu)\) and
\[
\mu\left(\{x : \psi([a_n]_{n \geq 1})(x) > t\}\right) \leq \frac{C}{t} \|[a_n]_{n \geq 1}\|_{\ell^1}, \quad \forall t > 0, \quad \forall [a_n]_{n \geq 1} \in \ell^1.
\]
We consider the \(\ell^p\) space as the space \(L^p(N, \text{card})\). By the Lemma 1.1.6. there exists a constant \(M(p) > 0\) such that
\[
\|\psi([a_n]_{n \geq 1})\|_{L^p} \leq M(p) \|[a_n]_{n \geq 1}\|_{\ell^p}.
\]
This implies that
\[
\|\phi(T)\|_{L^p} \leq M(p) \|T\|_{S_p}.
\]
So, \(\phi\) maps \(S_p\) boundedly into \(L^p(X, \mu)\), for every \(p \in (1, +\infty)\). \(\blacksquare\)

1.2 Spaces of Analytic Functions

We shall assume that the reader is familiar with the theory of Hardy spaces on the unit disk and we refer to \[5\]. Although, we will state some basic results.

1.2.1 Hardy Spaces

Definition 1.2.1. We will denote by \(L^p(\mathbb{T})\), \(p > 0\) the \(L^p(\mathbb{T}, m)\) space, where \(dm = \frac{1}{2\pi} |dz|\) is the normalized Lebesgue measure of \(\mathbb{T}\).
Definition 1.2.2. For \( p \in (0, +\infty) \) and \( f \in H(\mathbb{D}) \) we define the functions

\[
M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}} = \|f\|_{L_p^p(\mathbb{T})}, \quad r \in [0, 1) \quad \text{and} \\
M_\infty(r, f) = \sup_{\theta \in [0, 2\pi]} |f(re^{i\theta})| = \sup_{D(0, r)} |f(z)|, \quad r \in [0, 1).
\]

By the subharmonicity of the function \(|f(z)|^p\) it is easy to prove that \( M_p(r, f) \) is an increasing function of \( r \) and as a consequence

\[ \sup_{r \in [0, 1)} M_p(r, f) = \lim_{r \to 1^-} M_p(r, f). \]

The Hardy spaces \( H^p \), \( p > 0 \) are defined as

\[
H^p = \left\{ f \in H(\mathbb{D}) : \|f\|_{H^p} = \sup_{r \in (0, 1)} M_p(r, f) < +\infty \right\} \quad \text{and} \\
H^\infty = \left\{ f \in H(\mathbb{D}) : \|f\|_{H^\infty} = \sup_{\mathbb{D}} |f(z)| < +\infty \right\}.
\]

For \( p \in [1, +\infty) \), \( (H^p, \|\cdot\|_{H^p}) \) is a Banach space and for \( p = 2 \) the space \( H^2 \) is a separable Hilbert space with inner product

\[
\langle f, g \rangle = \sum_{n \geq 0} \hat{f}(n) \overline{\hat{g}(n)} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta.
\]

The point evaluation linear functionals on \( H^2 \) are bounded and the reproducing kernel at a point \( z \in \mathbb{D} \) has the form

\[
K_z(w) = \frac{1}{1 - \overline{z}w}.
\]

Proposition 1.2.1. Let \( f \in H^p \), \( p > 0 \), then for every \( z \in \mathbb{D} \)

\[
|f(z)| \leq \frac{\|f\|_{H^p}}{(1 - |z|^2)^{\frac{1}{p}}}.
\]

Theorem 1.2.1. (Fatou) Let \( f \in H^p \), \( p > 0 \), then the radial limits exist almost everywhere on \( \mathbb{T} \) and we denote the boundary values of \( f \) by \( f(e^{i\theta}) \), \( \theta \in [0, 2\pi] \)

\[
\lim_{r \to 1^-} f(re^{i\theta}) = f(e^{i\theta}).
\]

Moreover,

\[
\|f\|_{H^p} = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}.
\]
Theorem 1.2.2. (M. and F. Riesz) Let $f \in H^p$, $p > 0$, such that $f(e^{i\theta}) = 0$ on a subset of $\mathbb{T}$ with positive measure. Then, $f \equiv 0$.

Theorem 1.2.3. (F. Riesz) Let $f \in H^p$, $p > 0$, $f \neq 0$ and \{z_1, z_2, z_3, \ldots\} be the sequence of non-zero zeroes of $f$ counting the multiplicity. Then,

$$\sum_{n \geq 1} (1 - |z_n|) < +\infty$$

and the Blaschke product $B(z) = \prod_{n \geq 1} \frac{|z_n|}{z_n - z}$ converges.

There exists a function $g \in H^p$, $g(z) \neq 0$, $\forall z \in \mathbb{D}$ such that $f(z) = z^m B(z) g(z)$ and $\|g\|_{H^p} = \|f\|_{H^p}$, where $m \in \mathbb{N} \cup \{0\}$ is the multiplicity of 0 as a zero of $f$.

Theorem 1.2.4. If we identify every function $f \in H^p$, $p \in [1, +\infty]$ with its boundary-value function $f(e^{i\theta}) \in L^p(\mathbb{T})$, then

$$H^p = \{f \in L^p(\mathbb{T}) : \hat{f}(n) = 0, \forall n < 0\}.$$

For $p \in (1, +\infty)$, the Riesz projections $P_+$, $P_-$ are bounded, where

$$P_+ : L^p(\mathbb{T}) \to H^p, \quad P_+(f) = \sum_{n \geq 0} \hat{f}(n) e^{in\theta},$$

$$P_- : L^p(\mathbb{T}) \to H^p, \quad P_-(f) = \sum_{n = 1} \hat{f}(-n) e^{-in\theta} \quad \text{and}$$

$$H^p := \{f \in L^p(\mathbb{T}) : \hat{f}(n) = 0, \forall n \geq 0\}.$$

Corollary 1.2.1. For every $p \in (1, +\infty)$ and every function $f \in L^p(\mathbb{T})$, the partial sums of $f$ converges to $f$ in $L^p(\mathbb{T})$

$$S_n[f](e^{i\theta}) := \sum_{k=-n}^{n} \hat{f}(k) e^{ik\theta} \to f.$$

### 1.2.2 BMO and VMO Spaces

Definition 1.2.3. Let $I$ be an interval of $\mathbb{T}$ and $f \in L^1(\mathbb{T})$, we denote by $m_I(f)$ the integral mean of the function $f$ over the arc $I$

$$m_I(f) = \frac{1}{|I|} \int_I f(e^{i\theta}) d\theta,$$

where $|I| = \int_I d\theta$.

The quantity $m_I(|f - m_I(f)|)$ called the mean oscillation of $f$ over $I$

$$m_I(|f - m_I(f)|) = \frac{1}{|I|} \int_I \left| f(e^{i\theta}) - m_I(f) \right| d\theta.$$

The BMO space of functions of bounded mean oscillation is defined by

$$BMO = \left\{ f \in L^1(\mathbb{T}) : \|f\|_0 := \sup_{I \text{ interval of } \mathbb{T}} \{m_I(|f - m_I(f)|)\} < +\infty \right\}.$$
Proposition 1.2.2. Let \( f \in L^1(\mathbb{T}) \) and \( I \) be an interval of \( \mathbb{T} \). Then, for every \( x \in \mathbb{C} \)

\[
\frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)| \, d\theta \leq \frac{2}{|I|} \int_I |f(e^{i\theta}) - x| \, d\theta.
\]

Proof.

\[
\frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)| \, d\theta \leq \frac{1}{|I|} \int_I |f(e^{i\theta}) - x| \, d\theta + |x - m_I(f)| \leq \frac{2}{|I|} \int_I |f(e^{i\theta}) - x| \, d\theta.
\]

\( \blacksquare \)

Proposition 1.2.3. \( L^\infty(\mathbb{T}) \subset BMO \) and \( \|f\|_0 \leq \|f\|_{L^\infty}, \forall f \in L^\infty(\mathbb{T}). \)

Proof. Suppose \( I \) is an arbitrary interval of \( \mathbb{T} \), by the Cauchy-Schwarz inequality

\[
\frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)| \, d\theta \leq \left( \frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)|^2 \, d\theta \right)^{\frac{1}{2}}
\]

\[
= \left( \frac{1}{|I|} \int_I |f(e^{i\theta})|^2 \, d\theta - |m_I(f)|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \|f\|_{L^\infty}.
\]

\( \blacksquare \)

Remark. \( L^\infty(\mathbb{T}) \not\subset BMO. \)

Definition 1.2.4. We define a norm in the \( BMO \) space as

\[
\|f\|_{BMO} = |m_{\mathbb{T}}(f)| + \|f\|_0 = \left| \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \, d\theta \right| + \|f\|_0, \ f \in L^1(\mathbb{T}).
\]

It is easy to prove that \( \|f\|_0 = 0 \) if and only if \( f \) is constant on \( \mathbb{T} \) and that \( \cdot \|_{BMO} \) is actually a norm.

Lemma 1.2.1. Let \( f \in L^1(\mathbb{T}) \), then \( \|f\|_{L^1} \leq \|f\|_{BMO}. \)

Proof.

\[
\|f\|_{L^1} = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta}) - m_{\mathbb{T}}(f)| \, d\theta + |m_{\mathbb{T}}(f)| \leq \|f\|_{BMO}.
\]

\( \blacksquare \)
Theorem 1.2.5. \((BMO, \|\cdot\|_{BMO})\) is a Banach space.

**Proof.** Suppose \(\{f_n\}_{n \geq 1}\) is a Cauchy sequence in \(BMO\), by the definition of the norm and the Lemma 1.2.1. we obtain that \(\{m_I(f_n)\}_{n \geq 1}\) is a Cauchy sequence which converges to a complex number \(a \in \mathbb{C}\) and \(\{f_n\}_{n \geq 1}\) is a Cauchy sequence in \(L^1(\mathbb{T})\) which converges to an integrable function \(f\). By dominated convergence theorem follows that

\[
m_I(f_n) \to m_I(f).
\]

We will prove that \(f_n \to f\) in \(BMO\). By Fatou’s lemma, for every interval \(I \subset \mathbb{T}\)

\[
\frac{1}{|I|} \int_I |f_n - f - m_I(f) + m_I(f)|d\theta \leq \limsup_{m \to +\infty} \|f_n - f_m\|_0, \quad \|f_n - f\|_0 \leq \limsup_{m \to +\infty} \|f_n - f_m\|_0.
\]

This implies that \(f_n \to f \in BMO\) and \((BMO, \|\cdot\|_{BMO})\) is a Banach space. ■

**Definition 1.2.5.** The \(VMO\) space of functions of vanishing mean oscillation is defined by

\[
VMO = \left\{ f \in L^1(\mathbb{T}) : \lim_{|I| \to 0} \frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)| d\theta = 0 \right\}.
\]

**Proposition 1.2.4.** \(VMO\) is a closed subspace of \(BMO\).

**Proof.** Suppose \(f \in VMO\), for every \(\epsilon > 0\) there exists a \(\delta > 0\) such that

\[
\frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)| d\theta < \epsilon, \text{ if } |I| < \delta.
\]

(1.13)

We observe that

\[
\frac{1}{|I|} \int_I |f(e^{i\theta}) - m_I(f)| d\theta \leq \frac{2}{\delta} \|f\|_{L^1(\mathbb{T})}, \text{ if } |I| \geq \delta.
\]

(1.14)

By (1.13) and (1.14) follows that \(f \in BMO\) and as a consequence

\[VMO \subset BMO.\]

It remains to prove that \(VMO\) is closed in \(BMO\). Let \(\{f_n\}_{n \geq 1} \subset VMO\) be a sequence such that \(f_n \to f\) in \(BMO\). Then, for every \(\epsilon > 0\) there exists a \(N \in \mathbb{N}\) such that

\[
\|f_n - f\|_0 \leq \epsilon, \text{ } \forall n \geq N.
\]
This implies that
\[
\limsup_{|I| \to 0} \left( \frac{1}{|I|} \int_I |f - m_I(f)|d\theta \right) \leq \|f_N - f\|_0 + \limsup_{|I| \to 0} \left( \frac{1}{|I|} \int_I |f_N(e^{i\theta}) - m_I(f_N)|d\theta \right) \leq \epsilon,
\]
\[
\lim_{|I| \to 0} \frac{1}{|I|} \int_I f(e^{i\theta}) - m_I(f) \, d\theta = 0.
\]

Follows that \( f \in VMO \) and we conclude that \( VMO \) is a closed subspace of \( BMO \).

\[\text{Lemma 1.2.2.} \quad \overline{C(\mathbb{T})} \subset VMO.\]

**Proof.** \( \mathbb{T} \) is compact and as a consequence every continuous function on \( \mathbb{T} \) is uniformly continuous, the proof follows from this.

**Theorem 1.2.6.** Let \( f \in L^1(\mathbb{T}) \). Then, \( f \in VMO \) if and only if
\[
\lim_{r \to 1^-} \|u_r - f\|_{BMO} = 0,
\]
where \( u_r(e^{i\theta}) = u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)f(e^{it})dt \) is the Poisson integral of \( f \).

**Proof.** We assume first that \( \lim_{r \to 1^-} \|u_r - f\|_{BMO} = 0 \). We observe that \( u_r \in C(\mathbb{T}) \), for every \( r \in (0, 1) \) and by the Lemma 1.2.2. follows that \( f \in VMO \).

Conversely, we assume that \( f \in VMO \), by Fubini’s theorem for an arbitrary interval \( I \subset \mathbb{T} \)
\[
m_I(u_r - f) = \frac{1}{|I|} \int_I \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)f(e^{it})dt - f(e^{i\theta})d\theta
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t)\frac{1}{|I|} \int_I f(e^{i(\theta-t)}) - f(e^{i\theta})d\theta dt
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t)m_I(T(t)(f) - f)dt,
\]
where \( T_t(f(z)) := f(e^{-it}z) \).
By Fubini's theorem and (1.15)

\[
\frac{1}{|I|} \int_{I} |u_r - f - m_I(u_r - f)| \, dx = \frac{1}{|I|} \int_{I} \left| u_r - f - \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) m_I(T_t(f) - f) \, dt \right| \, dx
\]

\[
\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) \| T_t(f) - f \|_0 \, dt
\]

\[
\leq \frac{\| f \|_0}{\pi} \int_{0 < \delta < |t| < \pi} P_r(t) dt + \sup_{|t| \leq \delta} (\| T_t(f) - f \|_0), \quad \forall \delta \in (0, \pi).
\]

This implies that

\[
\limsup_{r \to 1} \| u_r - f \|_0 \leq \limsup_{t \to 0} \| T_t - f \|_0.
\]

We observe that \( m_I(T_t(f) - f) = m_I(u_r - f) = 0 \), it is sufficient to prove that

\[
\limsup_{t \to 0} \| T_t(f) - f \|_0 = 0.
\]

For every \( \epsilon > 0 \) there exists a \( \delta \in (0, \epsilon) \) such that

\[
\frac{1}{|I|} \int_{I} \left| f(e^{i\theta}) - m_I(f) \right| \, d\theta < \epsilon, \quad \forall |I| < \delta.
\]

By the dominated convergence theorem, there exists a \( \delta_1 \in (0, \delta) \) such that

\[
\| T_t(f) - f \|_{L^1} < \delta^2, \quad \forall t < \delta_1.
\]

If \( |I| \leq \delta_1 \),

\[
\frac{1}{|I|} \int_{I} |T_t(f) - f - m_I(T_t - f)| \, d\theta \leq \frac{1}{|I|} \int_{I} |T_t(f) - m_I(T_t(f))| \, d\theta
\]

\[
+ \frac{1}{|I|} \int_{I} |f - m_I(f)| \, d\theta \leq 2\epsilon. \quad (1.16)
\]

If \( |I| > \delta_1 \) and \( t < \delta_1 \),

\[
\frac{1}{|I|} \int_{I} |T_t(f) - f - m_I(T_t(f) - f)| \, d\theta \leq \frac{4\pi}{|I|} \| T_t(f) - f \|_{L^1} \leq 4\pi \epsilon. \quad (1.17)
\]

By (1.16) and (1.17)

\[
\lim_{t \to 0} \| T_t(f) - f \|_0 = 0,
\]

\[
\lim_{r \to 1} \| u_r - f \|_{BMO} = 0.
\]

\[\square\]
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Corollary 1.2.2. \( VMO = \overline{C(\mathbb{T})} \) in \( BMO \).

Definition 1.2.6. The Garcia seminorm \( \| \cdot \|_G \) in \( L^1(\mathbb{T}) \) is defined by

\[
\| f \|_G = \sup_{z \in \mathbb{D}} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta}) - u(z)| P(z, e^{i\theta}) d\theta \right),
\]

where \( u \) is the Poisson integral of \( f \) and \( P \) is the Poisson kernel.

Theorem 1.2.7. Let \( f \in L^1(\mathbb{T}) \), then

1. \( f \in BMO \) if and only if \( \| f \|_G < +\infty \).
2. \( f \in VMO \) if and only if \( \lim_{|z| \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta}) - u(z)| P(z, e^{i\theta}) d\theta = 0 \).

Moreover, there exist absolute constants \( C_1, C_2 > 0 \) such that for every \( g \in L^1(\mathbb{T}) \)

\[
C_1 \| g \|_0 \leq \| g \|_G \leq C_2 \| g \|_0.
\]

Proof. [10] \[\square\]

Theorem 1.2.8. \( BMO \) and \( VMO \) are conformally invariant spaces. Moreover, if \( f \in L^1(\mathbb{T}) \) and \( T \) is a holomorphic automorphism of the unit disk \( \mathbb{D} \), then

\[
\| f \|_G = \| f \circ T \|_G.
\]

Proof. Suppose \( T \) is a holomorphic automorphism of the unit disk. \( T \) can be written as

\[
T(z) = e^{it} T_w(z) = e^{it} \frac{w - z}{1 - \overline{w}z},
\]

where \( t \in \mathbb{R} \) and \( w \in \mathbb{D} \). By change of variables

\[
P[f](T(z)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |T(z)|^2}{|e^{i\theta} - T(z)|^2} f(e^{i\theta}) d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |T_w(z)|^2}{|e^{i\theta} - T_w(z)|^2} f(e^{i\theta}) d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |T_w(z)|^2}{|T_w(e^{i\theta}) - T_w(z)|^2} f(T_w(e^{i\theta})) T''_w(e^{i\theta}) d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} f \circ T(e^{i\theta}) d\theta.
\]
So,
\[ P[f](T(z)) = P[f \circ T](z). \] (1.18)

For an arbitrary point \( z \in \mathbb{D} \), there exists a point \( z' \in \mathbb{D} \) such that \( T(z') = z \), by (1.18)

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta}) - u(z)| P(z, e^{i\theta}) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta}) - P[f](z)| P(T(z'), e^{i\theta}) d\theta
\]
\[= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f \circ T(e^{i\theta}) - P[f \circ T](z')| P(z', e^{i\theta}) d\theta. \]

Thus,
\[ \|f\|_G = \|f \circ T\|_G. \]

\[\blacksquare\]

**Corollary 1.2.3.** Let \( f \in L^1(\mathbb{T}) \), then

1. \( \|f\|_G = \sup \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f \circ T - P[f \circ T](0)| d\theta : T \in Aut(\mathbb{D}) \right\} \).

2. \( f \in VMO \) if and only if \( \lim_{|a| \to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f \circ T - P[f \circ T_a](0)| d\theta = 0. \)

**Definition 1.2.7.** For a real valued function \( f \in L^1(\mathbb{T}) \) the conjugate function \( \tilde{f} \) is defined by

\[
\tilde{f}(z) = \frac{1}{2\pi} \int_{0}^{2\pi} Im \left( \frac{e^{it} + z}{e^{it} - z} \right) f(e^{it}) dt, \ z \in \mathbb{D}. \]

The non-tangential limits of \( \tilde{f} \) exist almost everywhere on \( \mathbb{T} \) and we will denote the boundary-valued function by \( \tilde{f} \). By linearity we can define the function \( \tilde{f} \) for every \( f \in L^1(\mathbb{T}) \) and it is easy to prove that

\[ \tilde{f} = -i(P_+ f - P_- f - \hat{f}(0)). \]

If \( f \in L^1(\mathbb{T}) \) is real valued, then \( \tilde{f} \) is the harmonic conjugate of \( P[f] \).

**Definition 1.2.8.** The \( BMOA \) space of holomorphic functions of bounded mean oscillation and the space \( VMOA \) of holomorphic functions of vanishing mean oscillation are defined by

\[ BMOA = BMO \cap H^1 \quad \text{and} \quad VMOA = VMO \cap H^1. \]
Theorem 1.2.9. *(Fefferman)*

\[ BMO = \{ \tilde{f} + g : f, g \in L^\infty(\mathbb{T}) \}. \]

**Proof.** [7] ■

Theorem 1.2.10. *(Sarason)*

\[ VMO = \{ \tilde{f} + g : f, g \in C(\mathbb{T}) \}. \]

**Proof.** [29] ■

Corollary 1.2.4.

\[ BMOA = P_+(L^\infty(\mathbb{T})) = BMO \cap H^2 \quad \text{and} \quad VMOA = P_+(C(\mathbb{T})) = VMO \cap H^2. \]

1.2.3 Bergman Spaces

**Definition 1.2.9.** Suppose \( a > -1 \) and \( p > 0 \), the weighted Bergman space \( A^p_a \) is defined by

\[ A^p_a = \left\{ f \in H(\mathbb{D}) : \| f \|_{A^p_a} := \left( \int_{\mathbb{D}} |f(z)|^p dA_a(z) \right)^{\frac{1}{p}} < +\infty \right\}, \]

where \( dA_a(z) = \frac{a+1}{\pi} (1 - |z|^2)^a dxdy, \ z = x + iy. \)

If \( a = 0 \) we denote by \( A^p \) the unweighted Bergman space and by \( dA(z) \) the unweighted normalized area measure. \( (A^p_a, \| \cdot \|_{A^p_a}) \) is a Banach space for \( a > -1 \), \( p \geq 1 \) and \( (A^2_a, \| \cdot \|_{A^2_a}) \) is a separable Hilbert space with inner product

\[ \langle f, g \rangle_{A^2_a} = \int_{\mathbb{D}} f(z) \overline{g(z)} dA_a(z) \]

and orthonormal basis \( \{ e_1, \ldots, e_n, \ldots \} \), where

\[ e_n = \frac{z^n}{\| z^n \|_{A^2_a}} = \left( \frac{\Gamma(n+a+2)}{n!\Gamma(a+2)} \right)^{\frac{1}{2}} z^n. \]

**Proposition 1.2.5.** Let \( a > -1 \), \( p > 0 \), then for every \( f \in A^p_a \) and for every \( z \in \mathbb{D} \)

\[ |f(z)| \leq \frac{\| f \|_{A^p_a}}{(1 - |z|^2)^{\frac{a+2}{p}}}. \]
Proof. For \( f \in A_a^p \), we observe that \(|f|^p\) is a subharmonic function

\[
|f(0)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta,
\]

\[
\int_0^1 r (1-r^2)^a |f(0)|^p dr \leq \frac{1}{2\pi} \iint_D |f(z)|^p (1-|z|^2)^a dxdy,
\]

\[
|f(0)|^p \leq \frac{a+1}{\pi} \iint_D |f(z)|^p (1-|z|^2)^a dxdy.
\]

For \( z \in \mathbb{D} \) and \( T_z(w) = \frac{z-w}{1-zw} \), it is easy to prove that \( f \circ T_z \in A_a^p \).

\[
|f \circ T_z(0)|^p \leq \frac{a+1}{\pi} \iint_D |f \circ T_z(w)|^p (1-|w|^2)^a dxdy, w = x + iy,
\]

\[
|f(z)|^p \leq \frac{a+1}{\pi} \iint_D |f(w)|^p \left( \frac{(1-|w|^2)(1-|z|^2)}{|1-zw|^2} \right)^a \left( \frac{1-|z|^2}{|1-zw|^2} \right)^2 dxdy, w = x + iy.
\]

We observe that \( f(w)(1-\overline{z}w)^{\frac{a+2}{p}} \in A_a^p \),

\[
|f(z)|^p \leq \frac{a+1}{\pi} \iint_D |f(w)|^p \left( \frac{(1-|w|^2)}{|1-\overline{z}w|^2} \right)^a dxdy, w = x + iy,
\]

\[
|f(z)| \leq \frac{\|f\|_{A_a^p}^p}{(1-|z|^2)^{\frac{a+2}{p}}}.
\]

\[\blacksquare\]

Corollary 1.2.5. Convergence in \( \left( A_a^p, \|\cdot\|_{A_a^p} \right) \) implies locally uniform convergence on \( \mathbb{D} \) and in case that \( p = 2 \), locally uniform and bounded convergence to 0 is equivalent to weak convergence to 0.

Proposition 1.2.6. The point evaluation linear functionals on \( A_a^2 \) are bounded and the reproducing kernel at a point \( z \in \mathbb{D} \) has the following form

\[
K_{z,a}(w) = \frac{1}{(1-\overline{z}w)^{a+2}}.
\]

Proof. Boundedness follows from the Proposition 1.2.5. and as a consequence there exists a function \( K_{z,a}(w) \in A_a^2 \) such that

\[
f(z) = \langle f, K_{z,a} \rangle_{A_a^2}.
\]
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\[ K_{z,a}(w) = \sum_{n=0}^{\infty} \langle K_{z,a}, e_n \rangle_{A^2} e_n = \sum_{n=0}^{\infty} e_n(z)e_n(w) = \sum_{n=0}^{\infty} \frac{\Gamma(n+a+2)}{n!\Gamma(a+2)} (\overline{z}w)^n = \frac{1}{(1-\overline{z}w)^{a+2}}. \]

By a standard density argument we obtain the following.

**Corollary 1.2.6.** Let \( f \in A^1_a, \ a > -1 \). Then, for every \( z \in \mathbb{D} \)

\[ f(z) = \int_{\mathbb{D}} f(w) \frac{1}{(1-z\overline{w})^{a+2}} dA_a(w). \]

**Proposition 1.2.7.** Let \( p \in (0,1), \ a > -1 \) and \( \gamma = \frac{2+a}{p} - 2 \). Then, for every \( f \in A^p_a \)

\[ \int_{\mathbb{D}} |f(z)| dA_{\gamma} \leq \frac{\gamma+1}{a+1} \left( \int_{\mathbb{D}} |f(z)|^p dA_a \right)^{\frac{1}{p}}. \]

**Proof.** By the Proposition 1.2.5.

\[ \int_{\mathbb{D}} |f(z)| dA_{\gamma} = \int_{\mathbb{D}} |f(z)|^p |f(z)|^{1-p} dA_{\gamma} \]

\[ \leq (\gamma + 1) \int_{\mathbb{D}} |f(z)|^p \frac{\|f\|_{A^p_a}^{1-p}}{(1-|z|)^{a+2}(1-|z|^2)^{1-p}} dA(z) \]

\[ \leq \frac{\gamma+1}{a+1} \|f\|_{A^p_a}^{1-p} \frac{\|f\|^p_{A^p_a}}{a+1} = \frac{\gamma+1}{a+1} \|f\|^p_{A^p_a}. \]

**Lemma 1.2.3.** (Hardy’s identity) Let \( f \) be a holomorphic function on \( \mathbb{D} \) and \( p > 0 \). Then,

\[ \frac{d}{dr} \left( r \frac{d}{dr} \int_{0}^{2\pi} |f(re^{i\theta})|^p d\theta \right) = p^2 r \int_{0}^{2\pi} \left| f(re^{i\theta}) \right|^{p-2} \left| f'(re^{i\theta}) \right|^2 d\theta. \]

**Proof.** One can prove that

\[ \left( \frac{d}{dr} \right)^2 |f(re^{i\theta})|^p + \left( \frac{d}{d\theta} \right)^2 |f(re^{i\theta})| = p^2 \left| f(re^{i\theta}) \right|^{p-2} \left| f'(re^{i\theta}) \right|^2. \]

By integration over \( \theta \) and differentiation under the integral sign we obtain that

\[ \frac{d}{dr} \left( r \frac{d}{dr} \int_{0}^{2\pi} |f(re^{i\theta})|^p d\theta \right) = p^2 r \int_{0}^{2\pi} \left| f(re^{i\theta}) \right|^{p-2} \left| f'(re^{i\theta}) \right|^2 d\theta. \]

**■**
**Proposition 1.2.8.** Let $p > 0$ and $f$ be a holomorphic and injective function on $\mathbb{D}$ such that $f(0) = 0$. Then,

$$
\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq 2\pi p \int_0^r \frac{M_\infty(t,f)^p}{t} dt.
$$

**Proof.** By Hardy’s identity

$$
\left( r \frac{d}{dr} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right) = p^2 \int_{D(0,r)} |f(z)|^{p-2} |f'(z)|^2 dxdy
$$

$$
= p^2 \int_{f(D(0,r))} |z|^{p-2} dxdy \leq p^2 \int_{D(0,M_\infty(r,f))} |z|^{p-2} dxdy
$$

$$
= 2\pi p^2 \int_0^{M_\infty(r,f)} x^{p-1} dx = 2\pi p M_\infty(r,f)^p,
$$

integrating over $r$

$$
\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq 2\pi p \int_0^r \frac{M_\infty(t,f)^p}{t} dt.
$$

\[\blacksquare\]

**Lemma 1.2.4.** Let $f$ be a holomorphic and injective function on $\mathbb{D}$ such that

$$
|f(z)| = O\left( \frac{1}{(1-|z|)^k} \right) \text{ as } |z| \to 1^-\text{, where } k \in [0,2].
$$

Then, for $p > \frac{1}{k}$

$$
\int_0^{2\pi} |f(re^{i\theta})|^p d\theta = O\left( \frac{1}{(1-r)^{kp-1}} \right) \text{ as } r \to 1^-.
$$

**Proof.** By the Proposition 1.2.8. there exists a radius $r_0 \in [0,1)$ and two constants $C_1$, $C_2 > 0$ such that

$$
\int_0^{2\pi} |f(re^{i\theta})|^p d\theta < C_1 \int_{r_0}^r M_\infty(t,f)^p dt < C_2 \int_{r_0}^r \frac{1}{(1-t)^{kp}} dt.
$$
For $p > \frac{1}{k}$,
\[
\int_0^{2\pi} |f(re^{i\theta})|^p d\theta = O\left(\frac{1}{(1-r)^kp-1}\right) \text{ as } r \to 1^-.\]

\[\blacksquare\]

Remark. By the Koebe distortion theorem [23], every injective function $f \in H(D)$ satisfies the following
\[
|f(z)| = O\left(\frac{1}{(1-|z|)^2}\right) \text{ as } |z| \to 1^-.
\]

Proposition 1.2.9. Let $x > 0$ and $y > -1$. Then, there exists a constant $C > 0$ such that for every $z \in \mathbb{D}$
\[
\int_D \frac{(1-|w|^2)^y}{|1-z\bar{w}|^{2x+y}}dA(w) \leq C \frac{1}{(1-|z|^2)^x}.
\]

Proof. By the Lemma 1.2.4, there exists a constant $C_0 > 0$ such that
\[
\int_D \frac{(1-|w|^2)^y}{|1-z\bar{w}|^{2x+y}}dA(w) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{(1-|r|^2)^y}{|1-zre^{-i\theta}|^{2x+y}}rd\theta dr
\]
\[
= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{(1-|r|^2)^y}{|1-zr|^{2x+y}}rd\theta dr \leq C_0 \int_0^1 \frac{(1-|r|^2)^y}{|1-|zr|^2|^{1+2x+y}} dr
\]
\[
= C_0 \int_0^{|z|} \frac{(1-|r|^2)^y}{(1-|zr|)^{1+2x+y}} dr + C_0 \int_{|z|}^1 \frac{(1-|r|^2)^y}{(1-|zr|)^{1+2x+y}} dr
\]
\[
\leq 2C_0 \frac{1}{(1-|z|)^{1+2x+y}} \int_0^{|z|} (1-r)^y dr + 2C_0 (1-|z|)^{1+y} \int_{|z|}^1 \frac{1}{(1-|rz|)^{2+2x+y}} dr.
\]
\[
\leq C \frac{1}{(1-|z|^2)^x}.
\]

\[\blacksquare\]

Lemma 1.2.5. Let $f \in A_a^1$, $a > -1$ and $n \in \mathbb{N}$. Then,
\[
f^{(n)}(z) = (a+2)\cdots(a+n+1) \int_D f(w) \frac{\bar{w}^n}{(1-z\bar{w})^{a+2+n}} dA_a(w).
\]
Proof. The proof follows from the Corollary 1.2.6. by differentiation under the integral sign. ■

Lemma 1.2.6. Let \( f \in A^p_a, p \geq 1, a > -1 \) such that \( f^{(n)} \in A^p_{a+pn} \) and \( f^{(k)}(0) = 0 \) for every \( k = 0, \ldots, n-1 \). Then,

\[
f(z) = \frac{1}{(a + 1)(a + 2) \cdots (a + n)} \int_D \frac{(1 - |w|^2)^n f^{(n)}(w)}{w^n (1 - z|w|^{a+2})} dA_a(w).
\]

Proof. We consider the function

\[
g(z) = \frac{1}{(a + 1)(a + 2) \cdots (a + n)} \int_D \frac{(1 - |w|^2)^n f^{(n)}(w)}{w^n (1 - z|w|^{a+2})} dA_a(w).
\]

By differentiation under the integral sign and the Corollary 1.2.6.

\[
g^{(n)}(z) = \int_D \frac{f^{(n)}(w)}{(1 - z|w|^{a+n+2})} dA_{a+n}(w) = f^{(n)}(z).
\]

We observe that

\[
f(0) = f'(0) = \ldots = f^{(n-1)}(0) = g(0) = g'(0) = \ldots = g^{(n-1)}(0) = 0.
\]

The claim follows. ■

Lemma 1.2.7. (Schur test) Suppose \((X, \mu)\) is a \(\sigma\)-finite measurable space and \(H(x, y)\) is a non-negative measurable function on \(X \times X\). For \(p > 1, \frac{1}{p} + \frac{1}{q} = 1\), if there exists a positive measurable function \(h\) on \(X\) such that

1. \(\int_X H(x, y)h^p(x) d\mu(x) \leq C_1 h^p(y)\), for almost every \(y \in X\).

2. \(\int_X H(x, y)h^q(y) d\mu(y) \leq C_2 h^q(x)\), for almost every \(x \in X\).

Then, the integral operator \(T(f) = \int_X H(x, y)f(y) d\mu(y)\) is bounded on \(L^p(X, d\mu)\) and \(\|T\| \leq C_1^{\frac{1}{p}} C_2^{\frac{1}{q}}\).

Proof. Let an arbitrary \(f \in L^p(X, d\mu)\), then applying Hölder’s inequality

\[
|T(f)| \leq \int_X H(x, y)|f(y)| d\mu(y) = \int_X H(x, y) \frac{h(y)}{h(y)} |f(y)| d\mu(y)
\]

\[
\leq \left( \int_X H(x, y) h(y)^q d\mu(y) \right)^{\frac{1}{q}} \left( \int_X H(x, y) h(y)^{-p} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}}
\]

\[
\leq C_1^{\frac{1}{p}} h(x) \left( \int_X H(x, y) h(y)^{-p} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}}.
\]
By Tonelli’s theorem
\[
\int_X |T(f)|^p d\mu(x) \leq C_{\frac{p}{q}}^p \int_X h^p(x) \int_X H(x, y)h(y)^{-p} |f(y)|^p d\mu(y)d\mu(x)
\]
\[
= C_{\frac{p}{q}}^p \int_X h(y)^{-p} |f(y)|^p \int_X H(x, y)h^p(x)d\mu(x)d\mu(y),
\]
\[
\leq C_{\frac{p}{q}}^p C_2 \int_X |f(y)|^p d\mu(y).
\]

We conclude that \(T\) is bounded on \(L^p(X, d\mu)\) and \(\|T\| \leq C_{\frac{p}{q}}^p C_2^\frac{1}{q}\). □

**Theorem 1.2.11.** Let \(p > 0, a > -1\) and \(n \in \mathbb{N}\). Then, \(f \in A_a^p\) if and only if \(f^{(n)} \in A_{a+n}^p\).

**Proof.** We assume first that \(f \in A_a^p\), by the Proposition 1.2.5. \(f \in A_1^\gamma\) for \(\gamma\) sufficiently large and as a consequence
\[
f(z) = (\gamma + 1) \int_{\mathbb{D}} f(w) \frac{(1 - |w|^2)^{\gamma}}{(1 - z \overline{w})\gamma+2} dA(w).
\]

By the Lemma 1.2.5.
\[
f^{(n)}(z) = (\gamma + 2) \ldots (\gamma + n + 1) \int_{\mathbb{D}} f(w) \frac{\overline{w}^n}{(1 - z \overline{w})\gamma+n+2} dA_\gamma(w).
\]

We define the function
\[
g(z) := \int_{\mathbb{D}} f(w) \frac{(1 - |w|^2)^{\gamma-a}}{(1 - z \overline{w})\gamma+n+2} (1 - |z|^2)^n dA_a(w).
\]

It is sufficient to prove that \(g \in L^p(dA_a)\).
\[
|g(z)| \leq \int_{\mathbb{D}} |f(w)| \frac{(1 - |w|^2)^{\gamma-a}}{|1 - z \overline{w}|\gamma+n+2} (1 - |z|^2)^n dA_a(w).
\]

For \(p = 1\), by Tonelli’s theorem and the Proposition 1.2.9. there exists a constant \(C > 0\), such that
\[
\int_{\mathbb{D}} |g(z)|dA_a(z) \leq \int_{\mathbb{D}} |f(w)| \frac{(1 - |w|^2)^{\gamma-a}}{|1 - z \overline{w}|\gamma+n+2} (1 - |z|^2)^n dA_a(w)dA_a(z)
\]
\[
= \int_{\mathbb{D}} |f(w)|(1 - |w|^2)^{\gamma-a} \int_{\mathbb{D}} \frac{|1 - |z|^2|^n}{|1 - z \overline{w}|\gamma+n+2} dA_a(z) dA_a(w)
\]
\[
\leq C \int_{\mathbb{D}} |f(w)|(1 - |w|^2)^n dA(w) < +\infty.
\]
For $p > 1$, it is sufficient to prove that the integral operator

$$T(h) = \int_D H(z, w) h(w) dA_a(w),$$

is bounded on $L^p(dA_a)$, where

$$H(z, w) = \frac{(1 - |w|^2)^{\gamma-a}}{|1 - zw|^{\gamma+n+2}(1 - |z|^2)^n}.$$

For $\gamma$ sufficiently large we can choose $\sigma > 0$ and a positive function $h(z) = \frac{1}{(1 - |z|^2)^{\sigma}}$ in order to use the Proposition 1.2.9. to obtain that the conditions of the Lemma 1.2.7. (Schur test) are satisfied

1. $\int_D H(z, w) h^p(z) dA_a(z) = \int_D \frac{1 - |z|^2)^{\gamma-a}}{|1 - zw|^{\gamma+n+2}} dA_a(z)(1 - |w|^2)^{\gamma-a} \leq C_1 h^p(w).$

2. $\int_D H(z, w) h^q(w) dA_a(w) = \int_D \frac{1 - |z|^2)^{\gamma-a}}{|1 - zw|^{\gamma+n+2}} dA_a(w)(1 - |z|^2)^n \leq C_2 h^q(z).$

By the Lemma 1.2.7. the integral operator $T$ is bounded.

For $p \in (0, 1)$, we choose $\gamma$ such that $\gamma = \frac{b+2}{p} - 2$ and $b > a > -1$. By the Proposition 1.2.7.

$$|g(z)|^p \leq \left(\frac{a+1}{\gamma+1}\right)^p (1 - |z|^2)^{pn} \left(\int_D \left|\frac{1}{|1 - zw|^{\gamma+n+2}} dA_a(z)\right|^p \right)^\frac{1}{p} \leq C_0 (1 - |z|^2)^{pn} \int_D |f(w)|^p \frac{1}{|1 - zw|^{\gamma+n+2}} dA_a(w).$$

By Tonelli’s theorem and the Proposition 1.2.9.

$$\int_D |g(z)|^p dA_a(z) \leq C_0 \int_D |f(w)|^p \int_D \frac{(1 - |z|^2)^{pn}}{|1 - zw|^{\gamma+n+2}} dA_a(z) dA_b(w) \leq C_0' \int_D |f(w)|^p dA_a(w) < +\infty,$$

where $C_0, C_0' > 0$ are constants. We conclude that $f^{(n)} \in A^p_{a+pn}$.

Conversely, we assume that $f^{(n)} \in A^p_{a+pn}$ or equivalently $g_0(z) := (1 - |z|^2)^n f^{(n)} \in L^p(dA_a)$. We observe that $f \in A^p_a$ if and only if
Let $f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k \in A^p_n$. So, we can assume that $f(0) = f'(0) = \ldots = f^{(n)}(0) = 0$ and by the Lemma 1.2.6.

$$f(z) = C(n, \gamma) \int_D \frac{(1-|w|^2)^n f^{(n)}(w)}{w^n(1-z\bar{w})^{r+2}} dA_{r}(w).$$

The function $\left| \frac{f^{(n)}}{w^n} \right|$ is bounded near 0, so there exists a constant $M > 0$ such that

$$|f(z)| \leq M \int_D |g_0(w)|(1-|w|^2)^{r-a} \frac{|1-z\bar{w}|^{r+2}}{|1-\bar{w}|^{r+2}} dA_{a}(w).$$

For $p = 1$, by Tonelli’s theorem and the Proposition 1.2.9. there exist constants $M'$ and $M''$ such that

$$\int_D |f(z)| dA_{a}(z) \leq M' \int_D |g_0(w)|(1-|w|^2)^{r-a} \frac{|1-z\bar{w}|^{r+2}}{|1-\bar{w}|^{r+2}} dA_{a}(z) dA(w) \leq M'' \int_D |g_0(w)| dA_{a}(w) < +\infty.$$

For $p > 1$, it is sufficient to prove that the integral operator

$$T_0(h) = \int_D H_0(z,w) h(w) dA_{a}(w),$$

is bounded on $L^p(dA_{a})$, where

$$H_0(z,w) = \frac{(1-|w|^2)^{r-a}}{|1-z\bar{w}|^{r+2}}.$$

For $\gamma$ sufficiently large we can choose $\sigma > 0$ and a positive function $h_0(z) = \frac{1}{(1-|z|^2)^{\sigma}}$ in order to use the Proposition 1.2.9. to obtain that the conditions of the Lemma 1.2.7. (Schur test) are satisfied

1. $\int_D H(z,w) h^p(z) dA_{a}(z) \leq C' \int_D \frac{(1-|z|^2)^{\sigma-pa}}{|1-z\bar{w}|^{r+2}} dA(z)(1-|w|^2)^{r-a} \leq C_1 h^p(w).$

2. $\int_D H(z,w) h^q(w) dA_{a}(w) \leq C'' \int_D \frac{(1-|w|^2)^{r-pa}}{|1-\bar{w}|^{r+2}} dA(w) \leq C_2 h^q(z).$

By the Lemma 1.2.7. the integral operator $T$ is bounded.
For $p \in (0, 1)$, we can choose $\gamma$ such that $\gamma = \frac{b+2}{p} - 2$, $b > a > -1$. By the Proposition 1.2.7. there exists a constant $M_0 > 0$ such that

$$|f(z)|^p \leq M^p \left( \frac{a+1}{\gamma + 1} \right)^p \left( \int_{\mathbb{D}} |g_0(w)|^p \frac{1}{|1-zw|^\gamma + 2} dA_\gamma(w) \right)^p$$

$$\leq M_0 \int_{\mathbb{D}} |g_0(w)|^p \frac{1}{|1-zw|^\gamma + 2p} dA_\gamma(w).$$

By Tonelli’s theorem and the Proposition 1.2.9. there exists a constant $M_1 > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^p dA_a(z) \leq M_0 \int_{\mathbb{D}} |g_0(w)|^p (1-|w|^2)^b \int_{\mathbb{D}} \frac{(1-|z|^2)^a}{|1-zw|^\gamma + 2p} dA(z) dA(w)$$

$$\leq M_1 \int_{\mathbb{D}} |g_0(w)|^p dA_a(w) < +\infty.$$ 

We conclude that $f \in A^p_a$. 

### 1.2.4 Besov Spaces

**Definition 1.2.10.** We will denote by $\mu$ the Borel measure on $\mathbb{D}$ defined as

$$d\mu(z) := \frac{dA(z)}{(1-|z|^2)^2}.$$ 

It is easy to prove using the Schwarz-Pick Lemma that the measure $\mu$ is invariant under holomorphic automorphisms of $\mathbb{D}$. In fact $4\pi \mu$ is the hyperbolic area measure of the unit disk.

**Theorem 1.2.12.** Let $f \in H(\mathbb{D})$, $p > 0$ and $n, m \in \mathbb{N}$ such that $pn, pm > 1$. Then,

$$\int_{\mathbb{D}} (1-|z|^2)^{pn} |f^{(n)}(z)|^p d\mu(z) < +\infty \text{ if and only if } \int_{\mathbb{D}} (1-|z|^2)^{pm} |f^{(m)}(z)|^p d\mu(z) < +\infty.$$ 

**Proof.** Without loss of generality we assume that $m > n$,

$$(pn-1) \int_{\mathbb{D}} (1-|z|^2)^{np} |f^{(n)}(z)|^p d\mu(z) = \left\| f^{(n)} \right\|_{A_p^{pn-2}}^p,$$

$$(pm-1) \int_{\mathbb{D}} (1-|z|^2)^{mp} |f^{(m)}(z)|^p d\mu(z) = \left\| f^{(m)} \right\|_{A_p^{pm-2}}^p.$$
By the Theorem 1.2.11.
\[ \|f^{(n)}\|_{A^{p}_{pn-2}} < +\infty \quad \text{if and only if} \quad \|f^{(m)}\|_{A^{p}_{pn-2 + p(m-n)}} = \|f^{(m)}\|_{A^{p}_{pm-2}} < +\infty. \]

\begin{definition}
For \( p > 0 \), the analytic Besov spaces are defined by
\[ B_{p} = \left\{ f \in H(D) : \int_{D} (1 - |z|^{2})^{pn} |f^{(n)}(z)|^{p} d\mu(z) < +\infty, \; pn > 1 \right\}. \]
For \( p > 1 \),
\[ B_{p} = \left\{ f \in H(D) : \|f\|_{B_{p}} := \left( \int_{D} (1 - |z|^{2})^{p} |f'(z)|^{p} d\mu(z) \right) \frac{1}{p} \right\} < +\infty \}
For \( p = 1 \),
\[ B_{1} = \left\{ f \in H(D) : \|f\|_{B_{1}} := \int_{D} |f''(z)| dA(z) < +\infty \right\}. \]
For \( p = 2 \), the Besov space \( B_{2} \) is equal to the Dirichlet space
\[ D = \left\{ f \in H(D) : \int_{D} |f'(z)|^{2} dA(z) < +\infty \right\}. \]
The Dirichlet space is a separable Hilbert space equipped with each of the following inner products
\[ \langle f, g \rangle_{D,1} = \langle f, g \rangle_{H^{2}} + \langle f', g' \rangle_{A^{2}}, \]
\[ \langle f, g \rangle_{D} = f(0)\overline{g(0)} + \langle f', g' \rangle_{A^{2}}. \]
\begin{definition}
The Bloch space of analytic functions is defined by
\[ B = \left\{ f \in H(D) : \|f\|_{B} := \sup_{D} \{(1 - |z|^{2})|f'(z)| \} < +\infty \right\}. \]
The Bloch space is a Banach space equipped with the norm
\[ \|f\| = |f(0)| + \|f\|_{B}. \]
Applying the Schwarz-Pick theorem it is easy to prove that for every \( f \in H(D) \) and for every holomorphic automorphism \( T \) of the unit disk \( \|f \circ T\|_{B} = \|f\|_{B}. \)
Theorem 1.2.13. Let \( f \in H(\mathbb{D}) \) and \( n, m \in \mathbb{N} \). Then,
\[
\sup_{\mathbb{D}} \left \{ (1 - |z|^2)^n \left | f^{(n)}(z) \right | \right \} < +\infty \quad \text{if and only if} \quad \sup_{\mathbb{D}} \left \{ (1 - |z|^2)^m \left | f^{(m)}(z) \right | \right \} < +\infty.
\]

Proof. Without loss of generality we can assume that \( m > n \) and that \( f(0) = \ldots = f^{(m)}(0) = 0 \). Suppose \( C_1 := \sup_{\mathbb{D}} \left \{ (1 - |z|^2)^n \left | f^{(n)}(z) \right | \right \} < +\infty \), we observe that \( f^{(n)}(z) \in A_n^1 \) and as a consequence
\[
f^{(n)}(z) = (n + 1) \int_{\mathbb{D}} f^{(n)}(w) \frac{(1 - |w|^2)^n}{(1 - z\bar{w})^{n+2}} dA(w).
\]

By the Lemma 1.2.5.
\[
f^{(m)}(z) = (\gamma + 1)(\gamma + 2)\ldots(\gamma + m - n + 1) \int_{\mathbb{D}} f^{(n)}(w) \frac{\bar{w}^{m-n}(1 - |w|^2)^n}{(1 - z\bar{w})^{m+2}} dA(w).
\]

By the Proposition 1.2.9. there exist a constant \( C > 0 \) such that
\[
(1 - |z|^2)^m \left | f^{(m)}(z) \right | \leq (1 - |z|^2)^n (\gamma + 1)\ldots(\gamma + m - n + 1) \int_{\mathbb{D}} \left | f^{(n)}(w) \right | \frac{(1 - |w|^2)^n}{|1 - z\bar{w}|^{m+2}} dA(w)
\]
\[
\leq C_1 C < +\infty.
\]

Thus,
\[
\sup_{\mathbb{D}} \left \{ (1 - |z|^2)^m \left | f^{(m)}(z) \right | \right \} < +\infty.
\]

Conversely, suppose \( C_2 := \sup_{\mathbb{D}} \left \{ (1 - |z|^2)^m \left | f^{(m)}(z) \right | \right \} < +\infty \). We observe that \( f^{(m)} \in L^1(dA_{n+m-n}) \) and by the Lemma 1.2.6. there exists a constant \( C' > 0 \) such that
\[
f^{(n)}(z) = C' \int_{\mathbb{D}} \frac{(1 - |w|^2)^{m-n} f^{(m)}(w)}{w^{m-n}(1 - z\bar{w})^{n+2}} dA_n(w).
\]

By the Proposition 1.2.9. and by the boundedness of the function \( \left | \frac{f^{(m)}(w)}{w^m} \right | \) near 0, there exist constants \( M, C'' > 0 \) such that
\[
(1 - |z|^2)^n f^{(n)}(z) \leq (1 - |z|^2)^n M \int_{\mathbb{D}} \frac{(1 - |w|^2)^{m-n} \left | f^{(m)}(w) \right |}{|1 - z\bar{w}|^{n+2}} dA_n(w)
\]
\[
\leq MC_2 C'' < +\infty.
\]

Thus,
\[
\sup_{\mathbb{D}} \left \{ (1 - |z|^2)^n \left | f^{(n)}(z) \right | \right \} < +\infty.
\]

\( \blacksquare \)
Corollary 1.2.7. Let $f \in H(D)$ and $n \in \mathbb{N}$. Then, $f \in \mathcal{B}$ if and only if

$$\sup_D \left\{ (1-|z|^2)^n \left| f^{(n)}(z) \right| \right\} < +\infty.$$ 

Theorem 1.2.14. For every $p, p' \in (0, +\infty)$ such that $p < p'$

$$\mathcal{B}_p \subset \mathcal{B}_{p'} \subset \mathcal{B}.$$ 

Proof. First we will prove that for every $p \in (0, +\infty)$

$$\mathcal{B}_p \subset \mathcal{B}.$$ 

For $f \in \mathcal{B}_p$ and $n \in \mathbb{N}$ with $np > 1$, there exists a constant $C_1 > 0$ such that

$$+\infty > C_1 > \int_D (1-|z|^2)^{np} \left| f^{(n)}(z) \right|^p d\mu(z)$$

$$\geq \frac{1}{2\pi} \int_0^{2\pi} \left| f^{(n)}(re^{i\theta}) \right|^p d\theta (1-r^2)^{np} dr$$

$$= \int_R^{1} M_p(r, f^{(n)}) d\theta (1-r^2)^{np} dr, \quad \forall R \in (0,1).$$

There exists a constant $C_2 > 0$ such that

$$M_p(R, f^{(n)}) < \frac{C_2}{(1-R)^{np}} \quad \forall R \in (0,1). \quad (1.19)$$

By the Hardy-Littlewood theorem for the integral means [5] and (1.19)

$$\sup_{z \in D} \left\{ (1-|z|^2)^n \left| f^{(n)}(z) \right| \right\} < +\infty.$$ 

By the Corollary 1.2.7. for every $p \in (0, +\infty)$

$$\mathcal{B}_p \subset \mathcal{B}.$$ 

Let $g \in \mathcal{B}_p$, $0 < p < p' < +\infty$ and $n \in \mathbb{N}$ such that $1 < np < np'$. Then,

$$\int_D (1-|z|^2)^{np'} \left| f^{(n)}(z) \right|^{p'} d\mu(z) = \int_D (1-|z|^2)^{np} \left| f^{(n)}(z) \right|^p (1-|z|^2)^{n(p'-p)} \left| f^{(n)}(z) \right|^{p'-p} d\mu(z)$$

$$\leq \sup_D \left\{ (1-|z|^2)^n \left| f^{(n)}(z) \right| \right\}^{p'-p} \int_D (1-|z|^2)^{np} \left| f^{(n)}(z) \right|^p d\mu(z)$$

$$< +\infty.$$
So, $g \in \mathcal{B}_{p'}$ and we conclude that
\[ \mathcal{B}_p \subset \mathcal{B}_{p'} \subset \mathcal{B}. \]

**Definition 1.2.13.** For $p \in [1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, we define the spaces of analytic functions
\[
\mathcal{B}^{-\frac{1}{p}}_q = \left\{ f \in H(D) : \|f\|_{\mathcal{B}^{-\frac{1}{p}}_q} = \left( \int_D (1 - |z|^2)^{2q-2} |f'(z)|^q \, dA(z) \right)^{\frac{1}{q}} \right\},
\]
\[
\mathcal{B}^{-1}_\infty = \left\{ f \in H(D) : \|f\|_{\mathcal{B}^{-1}_\infty} = \sup_D \{(1 - |z|^2)^2 |f'(z)| \} < +\infty \right\}.
\]

**Theorem 1.2.15.** Under the pairing
\[ \langle f, g \rangle = \sum_{n \geq 0} \hat{f}(n)\overline{g(n)}, \]
we have the following dualities:

1. For $p \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$
\[ \mathcal{B}^*_p \simeq \mathcal{B}^{-\frac{1}{p}}_q. \]

2. \[ \mathcal{B}^*_1 \simeq \mathcal{B}^{-1}_\infty. \]

**Proof.** \[22\]

### 1.3 Carleson Measures

**Definition 1.3.1.** For a point in the unit circle $e^{it} \in \mathbb{T}$ and a positive number $h > 0$, the Carleson sector at $e^{it}$ (1.1) is defined by
\[ S_h(e^{it}) = \left\{ re^{i\theta} \in \mathbb{D} : 1 - h \leq r < 1, |\theta - t| \leq h \right\}. \]

A positive Borel measure $\mu$ on $\mathbb{D}$ is a Carleson measure if
\[ \|\mu\|_c := \sup \left\{ \frac{\mu(S_h(e^{it}))}{h} : t \in [0, 2\pi), h > 0 \right\} < +\infty. \]
We observe that every Carleson measure is finite.

**Proposition 1.3.1.** Suppose $\mu$ is a positive Borel measure on $\mathbb{D}$ and $k > 0$. The measure $\mu$ is a Carleson measure if and only if

$$\sup \left\{ \frac{\mu(S_{k,h}(e^{it}))}{h} : t \in [0,2\pi), h > 0 \right\} < +\infty,$$

where

$$S_{k,h}(e^{it}) := \left\{ re^{i\theta} \in \mathbb{D} : 1 - h \leq r < 1, |\theta - t| \leq kh \right\}.$$

**Proof.** We assume first that $\mu$ is a Carleson measure and

$$C = \sup \left\{ \frac{\mu(S_{h}(e^{it}))}{h} : t \in [0,2\pi), h > 0 \right\} < +\infty.$$

It is easy to prove that $S_{k,h} \subset S_{kh}$, if $k \geq 1$ and $S_{k,h} \subset S_{h}$, if $k < 1$. Thus,

$$\mu(S_{k,h}) \leq Ch, \text{ if } k \geq 1 \text{ and } \mu(S_{k,h}) \leq Ch, \text{ if } k < 1.$$

$$\sup \left\{ \frac{\mu(S_{k,h}(e^{it}))}{h} : t \in [0,2\pi), h > 0 \right\} < +\infty.$$

Conversely, we observe that $S_{h} \subset S_{k,h}$, if $k \geq 1$ and $S_{h} \subset S_{k,h}$, if $k < 1$. The claim follows.
Corollary 1.3.1. Let $\mu$ be a positive Borel measure on $\mathbb{D}$. Then, $\mu$ is a Carleson measure if and only if \[ \sup_{I \text{ subarc of } T} \left( \frac{\mu(S_I)}{|I|} \right) < +\infty, \] where $S_I = \{ re^{i\theta} : 1-|I| \leq r < 1, \theta \in I \}$.

Theorem 1.3.1. (L. Carleson) Let $p > 0$ and $\mu$ be a positive Borel measure on $\mathbb{D}$. Then, $\mu$ is a Carleson measure if and only if there exists a constant $C > 0$ such that \[ \int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C \frac{2\pi}{\mu} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta, \quad \forall f \in H^p. \] Or equivalently if and only if the corresponding inclusion operator \[ I : H^p \to L^p(\mathbb{D}, \mu), \quad I(f) = f \] is bounded.

Moreover, there exist absolute constants $C_1, C_2 > 0$ such that \[ C_1 \| \mu \|_c \leq \| I \| \leq C_2 \| \mu \|_c. \]

Proof. [9]

Proposition 1.3.2. Let $\mu$ be a positive Borel measure on $\mathbb{D}$. Then, $\mu$ is a Carleson measure if and only if \[ \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-zw|^2} d\mu(w) < +\infty. \] Moreover, there exist absolute constants $C_1, C_2 > 0$ such that \[ C_1 \| \mu \|_c \leq \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-zw|^2} d\mu(w) \leq C_2 \| \mu \|_c. \]

Proof. Suppose $\mu$ is a Carleson measure, by Carleson's theorem there exists an absolute constant $C_2 > 0$ such that \[ \int_{\mathbb{D}} |f(w)|^2 d\mu(w) \leq C_2 \| \mu \|_c \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta, \quad \forall f \in H^2. \] We consider the normalized reproducing kernel at an arbitrary point $z \in \mathbb{D}$ \[ k_z(w) = \frac{\sqrt{1-|z|^2}}{1-zw}, \] \[ \int_{\mathbb{D}} |k_z(w)|^2 d\mu(w) \leq C_2 \| \mu \|_c \| k_z \|_{H^2}^2. \]
and as a consequence
\[
\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{z}w|^2} d\mu(w) \leq C_2 \|\mu\|_c < +\infty.
\]

Conversely, we define \( C := \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{z}w|^2} d\mu(w) < +\infty \) and we observe that \( \mu(\mathbb{D}) \leq C. \)

If \( h \geq \frac{1}{4} \), we have that
\[
\mu \left( S_h(e^{i\theta}) \right) \leq \mu(\mathbb{D}) \leq C \leq 4Ch, \quad \forall e^{i\theta} \in \mathbb{T}.
\]

If \( h < \frac{1}{4} \), for an arbitrary \( e^{i\theta} \in \mathbb{T} \) we choose the point \( z_0 = \left( 1 - \frac{h}{2} \right) e^{i\theta} \) and we obtain that
\[
\frac{1-|z_0|^2}{|1-\overline{z}_0w|^2} \geq \frac{C_0}{h}, \quad \forall w \in S_h(e^{i\theta}),
\]
where \( C_0 \) is an absolute constant.

\[
C \geq \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{z}_0w|^2} d\mu(w) \geq \int_{S_h(e^{i\theta})} \frac{1-|z|^2}{|1-\overline{z}_0w|^2} d\mu(w) \geq \frac{C_0}{h} \mu \left( S_h(e^{i\theta}) \right).
\]

\[\blacksquare\]

**Definition 1.3.2.** A finite positive Borel measure \( \mu \) on \( \mathbb{D} \) is a vanishing Carleson measure if
\[
\limsup_{h \to 0} \frac{\mu(S_h(z))}{h} = 0.
\]

Every vanishing Carleson measure is a Carleson measure on \( \mathbb{D} \).

**Theorem 1.3.2.** Let \( \mu \) be a finite positive Borel measure on \( \mathbb{D} \) and \( p > 0 \). Then, the following statements are equivalent:

1. \( \mu \) is a vanishing Carleson measure.

2. For every bounded sequence \( \{f_n\}_{n \geq 1} \subset H^p \) that converges locally uniformly to 0 holds that
\[
\lim_{n \to +\infty} \int_{\mathbb{D}} |f_n(w)|^p d\mu(w) = 0.
\]

3. \[
\lim_{|z| \to 1} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{z}w|^2} d\mu(w) = 0.
\]
Proof. 1. implies 2.

Let \( \{f_n\} \subset H^p \) be a bounded sequence in \( H^p \) that converges locally uniformly to 0. Then,

\[
\int_{D} |f_n(w)|^p \, d\mu(w) = \int_{D(0,r)} |f_n(w)|^p \, d\mu(w) + \int_{D} |f_n(w)|^p (d\mu - x_{D(0,r)}d\mu)(w).
\]

It is easy to prove, since \( \mu \) is a vanishing Carleson measure, that

\[
\lim_{r \to 1^-} \|d\mu - x_{D(0,r)}d\mu\|_c = 0.
\]

By Carleson’s theorem and the boundedness of the sequence \( \{f_n\}_{n \geq 1} \), for every \( \epsilon > 0 \) there exists a radius \( r \in (0, 1) \) such that

\[
\int_{D} |f_n(w)|^p (d\mu - x_{D(0,r)}d\mu)(w) \leq C \|f_n\|_{H^p} \|d\mu - x_{D(0,r)}d\mu\|_c \leq \epsilon,
\]

where \( C > 0 \) is an absolute constant.

By the locally uniformly convergence of \( \{f_n\}_{n \geq 1} \)

\[
\lim_{n \to +\infty} \int_{D(0,r)} |f_n(w)|^p \, d\mu(w) = 0,
\]

\[
\lim_{n \to +\infty} \int_{D} |f_n(w)|^p \, d\mu(w) = 0.
\]

2. implies 3.

We consider the family of functions \( \{f_z\} \), defined as

\[
f_z(w) = (k_z(w))^\frac{1}{p} = \left( \frac{1-|z|^2}{1-\bar{z}w} \right)^{\frac{1}{p}},
\]

where \( k_z \) is the normalized reproducing kernel at \( z \). The family \( \{f_z\} \) is bounded in \( H^p \) and converges locally uniformly to 0, as \( |z| \to 1^- \). This implies that

\[
\lim_{|z| \to 1^-} \int_{D} \frac{1-|z|^2}{|1-\bar{z}w|^2} \, d\mu(w) = 0.
\]

3. implies 1.
Suppose $h < \frac{1}{4}$, for an arbitrary $e^{i\theta} \in \mathbb{T}$ we choose the point $z_0 = (1 - \frac{h}{2})e^{i\theta}$ and we obtain that
\[
\frac{1 - |z_0|^2}{|1 - z_0w|^2} \geq \frac{C_0}{h}, \quad \forall w \in S_h(e^{i\theta}),
\]
where $C_0$ is an absolute constant.

Follows that
\[
\lim_{h \to 0} \sup_{|z| = 1 - \frac{h}{2}} \int_{D} \frac{1 - |z|^2}{|1 - zw|^2} d\mu(w) \leq \frac{1}{C_0} \lim_{h \to 0} \sup_{|z| = 1 - \frac{h}{2}} \int_{D} \frac{1 - |z|^2}{|1 - zw|^2} d\mu(w) = 0.
\]

**Corollary 1.3.2.** Suppose $\mu$ is a positive Borel measure on $\mathbb{D}$. The property of $\mu$ be a Carleson or a vanishing Carleson measure is independent of $p > 0$.

**Theorem 1.3.3.** Let $\mu$ be a Carleson measure on $\mathbb{D}$. Then, the inclusion operator $I : H^2 \to L^2(\mathbb{D}, \mu)$ is compact if and only if $\mu$ is a vanishing Carleson measure.

**Proof.** $I$ is compact if and only if for every sequence $\{f_n\}_{n \geq 1} \subseteq H^2$ that converges weakly to 0 holds that
\[
\|T(f_n)\|_{L^2(\mathbb{D}, \mu)} \to 0
\]
if and only if for every bounded sequence $\{f_n\}_{n \geq 1} \subseteq H^2$ that converges locally uniformly to 0 holds that
\[
\lim_{n \to +\infty} \int_{\mathbb{D}} |f_n(w)|^2 d\mu(w) = 0
\]
if and only if $\mu$ is a vanishing Carleson measure.

**Theorem 1.3.4.** Let $f$ be a holomorphic function on the unit disk $\mathbb{D}$. Then,

1. $f \in BMOA$ if and only if $|f'(z)|^2(1 - |z|^2)dA(z)$ is a Carleson measure.
2. $f \in VMOA$ if and only if $|f'(z)|^2(1 - |z|^2)dA(z)$ is a vanishing Carleson measure.
CHAPTER 1. PRELIMINARIES

**Theorem 1.3.5.** \( B_p \subset \text{VMOA}, \) for every \( p \in (0, +\infty) \).

**Proof.** By the Theorem 1.2.14. and the Theorem 1.3.4. it is sufficient to prove that for an arbitrary function \( f \in B_p \subset B_{2p} \), where \( p > 1 \), the corresponding measure \( d\mu_f = |f'(z)|^{2p} (1 - |z|^2) dA(z) \) is a vanishing Carleson measure.

Let \( e^{i\theta} \in \mathbb{T} \) and \( h > 0 \), then applying Hölder's inequality, \( \frac{1}{p} + \frac{1}{q} = 1 \)

\[
\mu_f(S_h(e^{i\theta})) = \int_{S_h(e^{i\theta})} \frac{|f'(z)|^{2p} (1 - |z|^2)^2}{(1 - |z|^2)^{\frac{2p}{p}}} \frac{1}{(1 - |z|^2)^{\frac{2}{q}-1}} dA(z)
\]

\[
\leq \left( \int_{S_h(e^{i\theta})} (1 - |z|^2)^{2p-2} |f'(z)|^{2p} dA(z) \right)^{\frac{1}{p}} \left( \int_{S_h(e^{i\theta})} \frac{dA(z)}{(1 - |z|^2)^{2q-2}} \right)^{\frac{1}{q}}.
\]

This implies that there exists a constant \( C_1 > 0 \) such that

\[
\sup_{z \in \mathbb{T}} \frac{\mu_f(S_h(z))}{h} \leq C_1 \sup_{z \in \mathbb{T}} \left( \int_{S_h(z)} (1 - |z|^2)^{2p-2} |f'(z)|^{2p} dA(z) \right)^{\frac{1}{p}}. \tag{1.20}
\]

We assumed that

\[
\int_{\mathbb{D}} (1 - |z|^2)^{2p-2} |f'(z)|^{2p} dA(z) < +\infty.
\]

By the dominated convergence theorem and (1.20)

\[
\lim_{h \to 0} \sup_{z \in \mathbb{T}} \frac{\mu_f(S_h(z))}{h} = 0.
\]

\[\blacksquare\]
Chapter 2

Additive Hankel Operators

Definition 2.0.1. An infinite matrix with complex entries that depend only on the sum of coordinates called (additive) Hankel matrix. Thus, an arbitrary Hankel matrix has the following form:

\[
\begin{pmatrix}
a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\
a_1 & a_2 & a_3 & a_4 & a_5 & \cdots \\
a_2 & a_3 & a_4 & a_5 & a_6 & \cdots \\
a_3 & a_4 & a_5 & a_6 & a_7 & \cdots \\
a_4 & a_5 & a_6 & a_7 & a_8 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Every Hankel matrix is constant on each secondary diagonal.

A sequence of complex numbers \(a = \{a_n\}_{n \geq 0}\) corresponds to a Hankel matrix, which we will denote by \(M_a = [a_{i+j}]_{i,j \geq 0}\).

For a function \(g \in L^1(\mathbb{T})\), the sequence of non-negative Fourier coefficients \(\{\hat{g}(n)\}_{n \geq 0}\) induces a Hankel matrix that we will denote by \(M_g\).

\[
M_g = \begin{pmatrix}
\hat{g}(0) & \hat{g}(1) & \hat{g}(2) & \hat{g}(3) & \cdots \\
\hat{g}(1) & \hat{g}(2) & \hat{g}(3) & \hat{g}(4) & \cdots \\
\hat{g}(2) & \hat{g}(3) & \hat{g}(4) & \hat{g}(5) & \cdots \\
\hat{g}(3) & \hat{g}(4) & \hat{g}(5) & \hat{g}(6) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Definition 2.0.2. Suppose \(a = \{a_n\}_{n \geq 0} \in \ell^2\) and \(M_a\) is the corresponding Hankel matrix. \(M_a\) induces an operator on the dense subspace \((c_{\infty}, \|\cdot\|_{\ell^2})\) of \((\ell^2, \|\cdot\|_{\ell^2})\), that we will denote by \(H_a\). For a sequence \(b = \{b_n\} \in c_{\infty}\)

\[
H_a(b) = c = \{c_n\}_{n \geq 0}, \quad \text{where} \quad c_n = \sum_{k \geq 0} a_{n+k} b_k
\]
or equivalently

\[
\begin{bmatrix}
   c_0 \\
   c_1 \\
   c_2 \\
   c_3 \\
   \vdots
\end{bmatrix} = \begin{bmatrix}
   a_0 & a_1 & a_2 & a_3 & \cdots \\
   a_1 & a_2 & a_3 & a_4 & \cdots \\
   a_2 & a_3 & a_4 & a_5 & \cdots \\
   a_3 & a_4 & a_5 & a_6 & \cdots \\
   \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \begin{bmatrix}
   b_0 \\
   b_1 \\
   b_2 \\
   b_3 \\
   \vdots
\end{bmatrix}
\]

It is easy to prove that \( H_a \) is well defined, there exists a \( n_0 \in \mathbb{N} \) such that \( b_n = 0, \ \forall n \geq n_0 \).

By Minkowski’s inequality

\[
\|c\|_{\ell^2} = \sqrt{\sum_{n=0}^{\infty} \sum_{k=0}^{n_0} a_{n+k} b_k^2} \leq \left( \sum_{n=0}^{n_0} \sum_{k=0}^{n_0} |a_{n+k} b_k|^2 \right)^{1/2} \leq \sum_{k=0}^{n_0} \left( \sum_{n=0}^{n_0} |a_{n+k} b_k|^2 \right)^{1/2} \leq (n_0 + 1) \|b\|_{\ell^\infty} \|a\|_{\ell^2} < +\infty.
\]

\( H_a \) is an (additive) Hankel operator on \( \ell^2 \) if it has an extension on \( \ell^2 \), that we will denote again by

\[ H_a : \ell^2 \to \ell^2. \]

**Definition 2.0.3.** Suppose \( g \in L^2(\mathbb{T}) \) and \( M_g \) is the corresponding Hankel matrix. \( M_g \) induces an operator on the dense subspace of polynomials in \( H^2 \), that we will denote by \( H_g \). For a complex polynomial \( f = \sum_{n=0}^{m} \hat{f}(n)e^{in\theta} \)

\[ H_g(f) = \sum_{n=0}^{\infty} c_n e^{in\theta} \in H^2, \text{ where } c_n = \sum_{k=0}^{\infty} \hat{g}(n+k)\hat{f}(k) \]

or equivalently

\[
\begin{bmatrix}
   c_0 \\
   c_1 \\
   c_2 \\
   c_3 \\
   \vdots
\end{bmatrix} = \begin{bmatrix}
   \hat{g}(0) & \hat{g}(1) & \hat{g}(2) & \hat{g}(3) & \cdots \\
   \hat{g}(1) & \hat{g}(2) & \hat{g}(3) & \hat{g}(4) & \cdots \\
   \hat{g}(2) & \hat{g}(3) & \hat{g}(4) & \hat{g}(5) & \cdots \\
   \hat{g}(3) & \hat{g}(4) & \hat{g}(5) & \hat{g}(6) & \cdots \\
   \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \begin{bmatrix}
   \hat{f}(0) \\
   \hat{f}(1) \\
   \hat{f}(2) \\
   \hat{f}(3) \\
   \vdots
\end{bmatrix}
\]

Similarly with the previous definition \( H_g \) is well defined. \( H_g \) is an (additive) Hankel operator on \( H^2 \) if it has an extension on \( H^2 \), that we will denote again by

\[ H_g : H^2 \to H^2. \] In this case the function \( g \) called a symbol of the Hankel operator \( H_g \).
A Hankel operator $H_g$ has many symbols, for example any two functions with the same analytic part or equivalently the same sequence of non-negative Fourier coefficients induce the same Hankel operator.

**Remark.** The definition of Hankel operators on $H^2$ is just a reformulation of the definition on $\ell^2$, induced by the isometric isomorphism between those spaces. So, we can work on either of those two spaces and obtain common results.

### 2.1 Boundedness

**Theorem 2.1.1. (Nehari)** Let $a = \{a_n\}_{n \geq 0} \in \ell^2$ and $H_a$ be the induced operator. Then, $H_a$ is a bounded Hankel operator if and only if there exists a bounded function $\hat{\psi} \in L^\infty(\mathbb{T})$ such that

$$\hat{\psi}(n) = a_n, \ \forall n \geq 0.$$  

Furthermore, if $H_a$ is a bounded Hankel operator, then

$$\|H_a\| = \min \{ \|\psi\|_{L^\infty} : \psi \in L^\infty(\mathbb{T}), \ \hat{\psi}(n) = a_n, \ \forall n \geq 0 \}.$$  

**Proof.** We assume first that there exists a function $\psi \in L^\infty(\mathbb{T})$ such that

$$\hat{\psi}(n) = a_n, \ \forall n \geq 0.$$  

For $b = \{b_n\}_{n \geq 0} \in c_\infty$ and $H_a(b) = c = \{c_n\}_{n \geq 0}$

$$\|c\|_{\ell^2}^2 = \sum_{n \geq 0} c_n \bar{c}_n = \sum_{n \geq 0} \sum_{k \geq 0} a_{n+k} b_k c_n = \sum_{n \geq 0} \sum_{m \geq n} a_m b_{m-n} \bar{c}_n = \sum_{m \geq 0} \sum_{n \geq 0} a_m b_{m-n} \bar{c}_n$$

$$= \sum_{m \geq 0} \hat{\psi}(m) \sum_{n \geq 0} b_{m-n} \bar{c}_n = \langle \psi, f \rangle_{H^2} = \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{i\theta}) \overline{f(e^{i\theta})g(e^{i\theta})} d\theta,$$

where

$$f(e^{i\theta}) = \sum_{n \geq 0} \overline{b_n} e^{i\theta n}, \quad g(e^{i\theta}) = \sum_{n \geq 0} c_n e^{i\theta}.$$  

By the Cauchy-Schwarz inequality

$$\|c\|_{\ell^2}^2 \leq \|\psi\|_{L^\infty} \|f\|_{H^2} \|g\|_{H^2},$$

$$\|H_a(b)\|_{\ell^2} \leq \|\psi\|_{L^\infty} \|b\|_{\ell^2}.$$  

By the density of $c_\infty$ in $\ell^2$ we obtain that $H_a$ is a bounded Hankel operator with norm

$$\|H_a\| \leq \|\psi\|_{L^\infty}.$$  (2.1)
CHAPTER 2. ADDITIVE HANKEL OPERATORS

Conversely, suppose $H_a$ is a bounded Hankel operator on $\ell^2$. We know that $L^1(\mathbb{T})^* = L^\infty(\mathbb{T})$ and we want to find a function $\psi \in L^\infty(\mathbb{T})$ such that

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{i\theta})e^{-in\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{-i\theta})e^{in\theta} d\theta, \forall n \geq 0.$$ 

We define the linear functional $L : H^1 \to \mathbb{C}$ as

$$L(f) = \sum_{n \geq 0} a_n \hat{f}(n).$$

By the canonical factorization theorem every function $f \in H^1$ can be written as $f = g_1 g_2$, where $g_1, g_2 \in H^2$ and $\|f\|_{H^1} = \|g_1\|_{H^2} \|g_2\|_{H^2}$.

$$|L(f)| = \left| \sum_{n \geq 0} a_n \hat{f}(n) \right| = \left| \sum_{n \geq 0} a_n \sum_{k=0}^{n} \hat{g}_1(k) \hat{g}_2(n-k) \right|$$

$$= \left| \sum_{n,m \geq 0} a_{n+m} \hat{g}_1(n) \hat{g}_2(m) \right| = \left| \langle H_a(\hat{g}_1), \hat{g}_2 \rangle \ell^2 \right|.$$

By the boundedness of $H_a$ and the Cauchy-Schwarz inequality

$$|L(f)| = \left| \langle H_a(\hat{g}_1), \hat{g}_2 \rangle \ell^2 \right| \leq \|H_a\| \|g_1\|_{H^2} \|g_2\|_{H^2} = \|H_a\| \|f\|_{H^1}.$$

$L$ is a bounded linear functional on $H^1$ with norm $\|L\| \leq \|H_a\|$. By the Hahn-Banach theorem $L$ has a bounded extension $\widetilde{L}$ on $L^1(\mathbb{T})$ and there exists a function $\psi \in L^\infty(\mathbb{T})$ such that

$$\widetilde{L}(f) = \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{-i\theta})f(e^{i\theta}) d\theta, f \in L^1(\mathbb{T}), \|\psi\|_{L^\infty} = \|\widetilde{L}\| \leq \|H_a\|. \quad (2.2)$$

We observe that

$$a_n = L(e^{in\theta}) = \widetilde{L}(e^{in\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{-i\theta})e^{in\theta} d\theta = \hat{\psi}(n), \forall n \geq 0.$$

This completes the proof and by (2.1) and (2.2) if $H_a$ is a bounded Hankel operator then

$$\|H_a\| = \min \left\{ \|\psi\|_{L^\infty} : \psi \in L^\infty(\mathbb{T}), \hat{\psi}(n) = a_n, \forall n \geq 0 \right\}.$$ 

$\blacksquare$
2.1. BOUNDEDNESS

We observe that in the above proof we worked with a linear functional on \( H^1 \) and we used the duality of \( L^1(\mathbb{T}) \). So, we can ask if there are similar characterizations associated with the space \( BMOA = (H^1)^* \).

**Theorem 2.1.2.** Let \( a = \{a_n\}_{n \geq 0} \in \ell^2 \) and \( H_a \) be the induced operator. Then, \( H_a \) is a bounded Hankel operator if and only if the function \( \phi = \sum_{n \geq 0} a_n z^n \) belongs to the \( BMOA \) space.

**Proof.** \( H_a \) is a bounded Hankel operator if and only if there exists a function \( \psi \in L^\infty(\mathbb{T}) \) such that \( \hat{\psi}(n) = a_n, \ \forall n \geq 0 \). We observe that

\[
\phi = P_+ \psi = \frac{1}{2} (i\hat{\psi} + \psi + \hat{\psi}(0))
\]

By the Theorem 1.2.9. \( H_a \) is a bounded Hankel operator if and only if \( \phi \in BMOA \). \( \blacksquare \)

The reformulation of the Nehari theorem for Hankel operators on the \( H^2 \) space is the following.

**Theorem 2.1.3. (Nehari)** Suppose \( g \in L^2(\mathbb{T}) \) and \( H_g \) is the induced operator. The following statements are equivalent:

- \( H_g \) is a bounded Hankel operator on \( H^2 \).
- There exists a bounded symbol \( \psi \in L^\infty \) of \( H_g \).
- \( P_+ g \in BMOA \).

Furthermore, if \( H_g \) is a bounded Hankel operator, then

\[
\|H_g\| = \min \{ \|\psi\|_{L^\infty} : \psi \text{ is a symbol of } H_g \}.
\]

**Corollary 2.1.1.** Let \( H_g \) be a bounded Hankel operator on \( H^2 \), then

\[
\|H_g\| = \min \{ \|g - f\|_{L^\infty} : f \in H^2 \}.
\]

The prototype of Hankel matrices is the Hilbert matrix

\[
\begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
The induced Hankel operator on $H^2$ called the Hilbert operator and denoted by $H$. By the Nehari theorem the Hilbert operator is a bounded Hankel operator and a bounded symbol is

$$\psi(e^{it}) = ie^{-it}(\pi - t), \ t \in [0, 2\pi), \ \|\psi\|_{L^\infty} = \pi.$$ 

The Hilbert operator can be written as an integral operator

$$H(f) = \int_{0}^{1} f(t) \frac{1}{1 - zt} dt.$$ 

The Hardy inequality is an immediate consequence of the canonical factorization theorem and the boundedness of the Hilbert operator

$$\sum_{n \geq 0} |\hat{f}(n)| \leq \pi \|f\|_{H^1}, \ \forall f \in H^1(\mathbb{D}).$$

### 2.2 Finite Rank

**Theorem 2.2.1. (Kronecker)** Let $a = \{a_n\}_{n \geq 0}$ be a sequence of complex numbers and $M_a$ be the induced Hankel matrix. Then, $M_a$ is of finite rank if and only if the function $\phi(z) = \sum_{n \geq 0} \frac{a_n}{z^{n+1}}$ is rational and in this case the rank of $M_a$ is equal to the number of poles of $\phi$.

**Proof.** Let $\text{rank}(M_a) = n \in \mathbb{N}$, then the first $n + 1$ columns of the matrix

$$M_a = \begin{pmatrix}
a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\
a_1 & a_2 & a_3 & a_4 & a_5 & \cdots \\
a_2 & a_3 & a_4 & a_5 & a_6 & \cdots \\
a_3 & a_4 & a_5 & a_6 & a_7 & \cdots \\
a_4 & a_5 & a_6 & a_7 & a_8 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

are linear depended or equivalently there exist $c_0, c_1, \ldots, c_n$, not all equal to zero, such that

$$c_0 \sum_{k \geq 0} a_k z^{-k-1} + \ldots + c_n \sum_{k \geq 0} a_{k+n} z^{-k-1} = 0.$$ 

We observe that $c_n \neq 0$, otherwise

$$\sum_{k \geq 0} a_{k+n} z^k \in \text{span}\left\{\sum_{k \geq 0} a_{k+i} z^k : i = 0, 1, \ldots, n-1\right\},$$
or equivalently
\[ S^*(n) \left( \sum_{k \geq 0} a_k z^k \right) = \lambda_0 \sum_{k \geq 0} a_k z^k + \lambda_1 S^* \left( \sum_{k \geq 0} a_k z^k \right) + \ldots + \lambda_{n-1} S^*(n-1) \left( \sum_{k \geq 0} a_k z^k \right), \quad \lambda_i \in \mathbb{R}, \]

where \( S^* \) is the backward shift operator \( S^* \left( \sum_{n \geq 0} \hat{f}(n) z^n \right) = \sum_{n \geq 0} \hat{f}(n+1) z^n \).

By induction, using the linearity of \( S^* \) we can prove that for every \( m \in \mathbb{N} \)
\[ S^{*(m)} \left( \sum_{k \geq 0} a_k z^k \right) \in \text{span} \left\{ S^{*(i)} \left( \sum_{k \geq 0} a_k z^k \right) : i = 0, \ldots, n-1 \right\}, \]

or equivalently that every column of the matrix \( M_a \) is a linear combination of the first \( n \) linear depended columns and as a consequence \( \text{rank}(M_a) \leq n - 1 \), contradiction.

\[
0 = \sum_{j=0}^{n} c_j \sum_{k \geq 0} a_k z^{j-k-1} = \sum_{j=0}^{n} c_j z^j \sum_{k \geq 0} a_k z^{j-k-1-j}
= c_0 \phi + \sum_{j=1}^{n} c_j z^j \left( \phi - \sum_{k=0}^{j-1} a_k z^{k-1} \right)
= \phi \sum_{j=0}^{n} c_j z^j - \sum_{j=0}^{n-1} b_j z^j,
\]

for some \( b_j \in \mathbb{C} \). Follows that
\[
\phi = \frac{\sum_{j=0}^{n-1} b_j z^j}{\sum_{j=0}^{n} c_j z^j}
\]

and since \( c_n \neq 0 \) we obtain that \( \phi \) is a rational function with number of poles \( N \) and
\[ N \leq \text{rank}(M_a) = n. \quad (2.3) \]

Conversely, if \( \phi \) is a rational function with number of poles \( N \), then can be written in the form
\[
\phi = \frac{\sum_{j=0}^{N-1} b_j z^j}{\sum_{j=0}^{N} c_j z^j},
\]

or equivalently
\[
2.2. \text{FINITE RANK}
\]

or equivalently
\[
\sum_{k \geq 0} a_k z^k = \lambda_0 \sum_{k \geq 0} a_k z^k + \lambda_1 S^* \left( \sum_{k \geq 0} a_k z^k \right) + \ldots + \lambda_{n-1} S^*(n-1) \left( \sum_{k \geq 0} a_k z^k \right), \quad \lambda_i \in \mathbb{R},
\]

where \( S^* \) is the backward shift operator \( S^* \left( \sum_{n \geq 0} \hat{f}(n) z^n \right) = \sum_{n \geq 0} \hat{f}(n+1) z^n \).

By induction, using the linearity of \( S^* \), we can prove that for every \( m \in \mathbb{N} \)
\[
S^{*(m)} \left( \sum_{k \geq 0} a_k z^k \right) \in \text{span} \{ S^{*(i)} \left( \sum_{k \geq 0} a_k z^k \right) : i = 0, \ldots, n-1 \},
\]

or equivalently that every column of the matrix \( M_a \) is a linear combination of the first \( n \) linear depended columns and as a consequence \( \text{rank}(M_a) \leq n - 1 \), contradiction.

\[
0 = \sum_{j=0}^{n} c_j \sum_{k \geq 0} a_k z^{j-k-1} = \sum_{j=0}^{n} c_j z^j \sum_{k \geq 0} a_k z^{j-k-1-j}
= c_0 \phi + \sum_{j=1}^{n} c_j z^j \left( \phi - \sum_{k=0}^{j-1} a_k z^{k-1} \right)
= \phi \sum_{j=0}^{n} c_j z^j - \sum_{j=0}^{n-1} b_j z^j,
\]

for some \( b_j \in \mathbb{C} \). Follows that
\[
\phi = \frac{\sum_{j=0}^{n-1} b_j z^j}{\sum_{j=0}^{n} c_j z^j}
\]

and since \( c_n \neq 0 \) we obtain that \( \phi \) is a rational function with number of poles \( N \) and
\[ N \leq \text{rank}(M_a) = n. \quad (2.3) \]

Conversely, if \( \phi \) is a rational function with number of poles \( N \), then can be written in the form
\[
\phi = \frac{\sum_{j=0}^{N-1} b_j z^j}{\sum_{j=0}^{N} c_j z^j},
\]
and trivially
\[
\sum_{k \geq 0} \sum_{j=0}^{N} a_k c_j z^{j-k-1} = \sum_{j=0}^{N-1} b_j z^j,
\]
\[
0 = \sum_{j=0}^{N} \sum_{k \geq j} a_k c_j z^{j-k-1},
\]
\[
0 = \sum_{j=0}^{N} \sum_{m \geq 0} a_{m+j} c_j z^{-m-1}.
\]
This implies that the first \(N + 1\) columns of the matrix \(M_a\) are linear depended.
We observe that \(c_N \neq 0\), otherwise the function \(\phi\) has number of poles less than \(N\). Every column of \(M_a\) is a linear combination of the first \(N + 1\) columns and as a consequence
\[
\text{rank}(M_a) \leq N.
\]
By (2.3) and (2.4) we obtain that if \(M_a\) is a Hankel matrix of finite rank, then the rank of \(M_a\) is equal to the number of poles of \(\phi\).

**Theorem 2.2.2. (Kronecker)** Let \(H_g\) be a Hankel operator on \(H^2\). Then, \(H_g\) is of finite rank if and only if the function \(\phi(z) = \sum_{n \geq 0} \hat{g}(n) z^n\) is rational. In this case, the rank of \(H_g\) is equal to the number of poles of \(\phi\).

**Corollary 2.2.1.** Every complex polynomial \(P\) induces a compact Hankel operator \(H_P\) on \(H^2\).

### 2.3 Compactness

**Lemma 2.3.1.** Let \(T : H^2 \to H^2\) be a compact operator and let \(S\) be the Shift operator acting on \(H^2\), i.e. \(S(f) = z f(z)\). Then,
\[
\lim_{n \to +\infty} \left\| T \circ S^{(n)} \right\| = 0.
\]

**Proof.** Since \(T\) is compact, there exists a sequence of bounded operators of finite rank \(\{T_m\}_{m \geq 0}\) that converges to \(T\). For every \(\epsilon > 0\) there exists an integer \(N\) such that
\[
\|T - T_m\| < \epsilon, \quad \forall m \geq N.
\]
We observe that the Shift operator \(S\) is a contraction and for \(m \geq N\)
\[
\left\| T \circ S^{(n)} \right\| \leq \left\| T \circ S^{(n)} - T_m \circ S^{(n)} \right\| + \left\| T_m \circ S^{(n)} \right\| \\
\leq \epsilon + \left\| T_m \circ S^{(n)} \right\| \\
= \epsilon + \left\| R_1 \circ S^{(n)} + \ldots + R_k \circ S^{(n)} \right\|,
\]
2.3. COMPACTNESS

where $R_i, i = 1, \ldots, k$ are rank-one operators. By the Riesz representation theorem there exist $g_i, h_i \in H^2, i = 1, \ldots, k$ such that

$$R_i(f) = \langle f, g_i \rangle h_i.$$ 

The adjoint operator of $S$ is the backward Shift operator $S^*$ and by the Cauchy-Schwarz inequality we have that

$$\left\| R_i \circ S^{(n)}(f) \right\|_{H^2} = \left\| \langle S^{(n)}(f), g_i \rangle \right\|_{H^2} = \left\| \langle f, S^*(g_i) \rangle \right\|_{H^2} \leq \|f\|_{H^2} \|h_i\|_{H^2} \left\| S^*(g_i) \right\|_{H^2} \leq \|f\|_{H^2} \|h_i\|_{H^2} \sqrt{\sum_{k \geq n} |\hat{g}(k)|^2}.$$ 

Follows that $\lim_{n \to +\infty} \left\| R_i \circ S^{(n)} \right\| = 0$ and as a consequence $\lim_{n \to +\infty} \left\| T \circ S^{(n)} \right\| = 0.$

**Lemma 2.3.2.** Let $g \in C(\mathbb{T}),$ then $\text{dist}(g, C(\mathbb{T}) \cap H_\infty^\infty) = \text{dist}(g, H_\infty^\infty).$

**Proof.** Trivially $\text{dist}(g, C(\mathbb{T}) \cap H_\infty^\infty) \geq \text{dist}(g, H_\infty^\infty).$

Suppose $f \in H_\infty^\infty$ and $P_D[f](z), P_D[\text{Reg}](z), P_D[\text{Img}](z)$ are the Poisson integrals of $f, \text{Reg}$ and $\text{Img}$ respectively

$$\|f - g\|_{L_\infty^\infty} = \|f - \text{Reg} - i\text{Img}\|_{L_\infty^\infty} \geq \lim_{r \to 1} \|P_D[f](rz) - P_D[\text{Reg}](rz) - iP_D[\text{Img}](rz)\|_{L_\infty^\infty} \geq \lim_{r \to 1} \|P_D[f](rz) - g(z)\|_{L_\infty^\infty} - \lim_{r \to 1} \|g(z) - P_D[\text{Reg}](rz) - iP_D[\text{Img}](rz)\|_{L_\infty^\infty} = \lim_{r \to 1} \|P_D[f](rz) - g(z)\|_{L_\infty^\infty} \geq \text{dist}(g, C(\mathbb{T}) \cap H_\infty^\infty).$$

Follows that $\text{dist}(g, C(\mathbb{T}) \cap H_\infty^\infty) = \text{dist}(g, H_\infty^\infty).$

**Definition 2.3.1.** Suppose $(X, \|\cdot\|_X)$ is a normed space and $M$ is a closed subspace of $X.$ The quotient space associated with $M$ is defined by $(X/M, \|\cdot\|_{X/M}),$ where $\|x + M\|_{X/M} = \text{dist}(x, M).$ It is easy to prove that if $X$ is a Banach space, then $X/M$ is also a Banach space.

**Lemma 2.3.3.** $H_\infty^\infty + C(\mathbb{T})$ is closed in $L_\infty^\infty(\mathbb{T}).$
Proof. We consider the map
\[ \phi : C(\mathbb{T})/H^\infty \cap C(\mathbb{T}) \to (H^\infty + C(\mathbb{T}))/H^\infty, \quad \phi(f + H^\infty \cap C(\mathbb{T})) = f + H^\infty. \]

By the isomorphism theorem for vector spaces \( \phi \) is an isomorphism. Furthermore,
\[
\| f + H^\infty \cap C(\mathbb{T}) \|_{C(\mathbb{T})/H^\infty \cap C(\mathbb{T})} = \text{dist}(f, H^\infty \cap C(\mathbb{T})) = \| \phi(f + H^\infty \cap C(\mathbb{T})) \|_{L^\infty(\mathbb{T})/H^\infty}.
\]

This implies that \( \phi \) is an isometric isomorphism and \((H^\infty + C(\mathbb{T}))/H^\infty\) is a closed subspace of \( L^\infty(\mathbb{T})/H^\infty \).

We consider the natural quotient map
\[ \psi : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})/H^\infty, \quad \psi(f) = f + H^\infty, \ f \in L^\infty(\mathbb{T}). \]

Clearly \( \psi \) is a bounded operator and as a consequence
\[
\psi^{-1}((H^\infty + C(\mathbb{T}))/H^\infty) = H^\infty + C(\mathbb{T}) \text{ is closed in } L^\infty(\mathbb{T}).
\]

Lemma 2.3.4. \( z^n H^\infty \subseteq H^\infty + C(\mathbb{T}) \), for every \( n \in \mathbb{N} \).

Proof. An arbitrary function \( z^n f = z^n \sum_{k>0} \hat{f}(-k)z^k \in z^n H^\infty \) can be written as a sum of a function with vanishing non-negative Fourier coefficients and a continuous one
\[
\begin{aligned}
z^n f &= z^n \sum_{k=1}^n \hat{f}(-k)z^k + \sum_{k>n} \hat{f}(-k)z^{k-n}, \quad \forall z \in \mathbb{T}.
\end{aligned}
\]

Theorem 2.3.1. (Hartman) Let \( g \in L^2(\mathbb{T}) \) and \( H_g \) be the induced operator. Then, \( H_g \) is a compact Hankel operator on \( H^2 \) if and only if there exists a continuous symbol \( \psi \in C(\mathbb{T}) \) of \( H_g \).

Proof. We suppose first that there exists a continuous symbol \( \psi \) of \( H_g \). By the Nehari theorem \( H_g \) is a bounded Hankel operator. Trigonometric polynomials are dense in \( C(\mathbb{T}) \), there exists a sequence of trigonometric polynomials \( \{P_n\}_{n \geq 1} \) that converges to \( \psi \) with respect to the supremum norm
\[
\| H_{P_n} - H_g \| = \| H_{P_n} - H_\psi \| = \| H_{(P_n - \psi)} \| \leq \| P_n - \psi \|_{L^\infty} \to 0.
\]

By Kronecker's theorem the sequence \( \{H_{P_n}\}_{n \geq 1} \) is a sequence of finite rank bounded operators that converges to \( H_g \). So, \( H_g \) is a compact Hankel operator.
Conversely, suppose $H_g$ is a compact Hankel operator. Without loss of generality we assume that $g \in L^\infty(\mathbb{T})$. By the Lemma 2.3.4.

\[
\|H_g \circ S^{(n)}\| = \|H_{\overline{z}^n g}\| = \min \left\{ \|\overline{z}^n g - f\|_{L^\infty} : f \in H^\infty \right\} = \min \left\{ \|g - z^n f\|_{L^\infty} : f \in H^\infty \right\} = \text{dist} \left( g, z^n H^\infty \right) \geq \text{dist} \left( g, H^\infty + C(\mathbb{T}) \right).
\]

Follows by the Lemma 2.3.1. that $\text{dist} \left( g, H^\infty + C(\mathbb{T}) \right) = 0$. We obtain that $g \in H^\infty + C(\mathbb{T})$, since $H^\infty + C(\mathbb{T})$ is closed in $L^\infty(\mathbb{T})$ and as a consequence there exists a continuous symbol $\psi \in C(\mathbb{T})$ of $H_g$.

**Corollary 2.3.1.** Suppose $g \in L^2(\mathbb{T})$ and $H_g$ is the induced operator. $H_g$ is a compact Hankel operator if and only if there exists a sequence of bounded Hankel operators of finite rank that converges to $H_g$.

**Theorem 2.3.2.** Let $g \in L^2(\mathbb{T})$ and $H_g$ be the induced operator. Then, $H_g$ is a compact Hankel operator if and only if the function $P_+ g$ belongs to the VMOA space.

**Proof.** $H_g$ is a compact Hankel operator if and only if there exists a continuous symbol $\psi \in C(\mathbb{T})$ of $H_g$. We observe that

\[
P_+ g = P_+ \psi = \frac{1}{2}(i \overline{\psi} + \psi + \hat{\psi}(0)).
\]

By the Theorem 1.2.10. $H_g$ is a compact Hankel operator if and only if $P_+ g \in \text{VMOA}$. ■

The reformulation of Hartman’s theorem for Hankel operators on the $\ell^2$ space is the following.

**Theorem 2.3.3. (Hartman)** Let $a = \{a_n\}_{n \geq 0} \in \ell^2$ and $H_a$ be the induced operator. Then, the following statements are equivalent:

- $H_a$ is a compact Hankel operator on $\ell^2$.
- There exists a function $\psi \in C(\mathbb{T})$ such that $\hat{\psi}(n) = a_n$, $\forall n \geq 0$.
- $\phi = \sum_{n \geq 0} a_n z^n$ belongs to the VMOA space.
2.4 Schatten Class

**Theorem 2.4.1.** Let \( f \in H(\mathbb{D}) \) and \( H_f \) be a Hankel operator on \( H^2 \). Then, \( H_f \) is a Hilbert-Schmidt operator if and only if \( f \) belongs to the Dirichlet space.

**Proof.** If \( f \in \mathcal{D} \), then by the Theorem 1.3.5. \( f \in VMOA \) and by Hartman’s theorem \( H_f \) is compact.

\[
\|H_f\|_{S_2}^2 = \sum_{n \geq 0} \|H_f(z^n)\|_{H^2}^2 = \sum_{n \geq 0} \sum_{k \geq n} |\hat{f}(k)|^2
\]

\[
= \sum_{k \geq 0} (k+1)|\hat{f}(k)|^2 = \|f\|_{H^2}^2 + \int_\mathbb{D} |f'(z)|^2 dA(z).
\]

\[\square\]

**Theorem 2.4.2.** Let \( f \in H(\mathbb{D}) \) and \( H_f \) be a Hankel operator on \( H^2 \). Then, \( H_f \) belongs to the trace class \( S_1 \) if and only if \( f \) belongs to the Besov space \( \mathcal{B}_1 \).

**Proof.** We assume first that \( f \in \mathcal{B}_1 \), by Hartman’s theorem \( H_f \) is compact. It is easy to prove that \( H_f \in S_1 \) if and only if \( H_{f-P} \in S_1 \), where \( P \) is a complex polynomial. Thus, we can assume that \( f(0) = f'(0) = f''(0) = f'''(0) = 0 \).

By the Corollary 1.2.6. integrating twice we obtain that

\[
f''(z) = 2 \int_\mathbb{D} \frac{1-|w|^2}{(1-ar{w}z)^3} f''(w)dA(w),
\]

\[
f(z) = \int_\mathbb{D} \frac{1-|w|^2}{(1-ar{w}z)^2} f''(w)dA(w).
\]  

(2.5)

Suppose \( g \in H^2 \), by (2.5) and Fubini’s theorem

\[
H_f(g) = P_+(f(z)g(\bar{z})) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})g(e^{-i\theta})}{1-ze^{-i\theta}} d\theta
\]

\[
= \int_\mathbb{D} \frac{1-|w|^2}{w^2} f''(w) \frac{1}{2\pi} \int_0^{2\pi} \frac{g(e^{-i\theta})}{1-e^{i\theta}w} \frac{1}{1-ze^{-i\theta}} d\theta dA(w).
\]

Thus,

\[
H_f(g) = \int_\mathbb{D} \frac{1-|w|^2}{w^2} f''(w) H_{K_w}(g(z))dA(w),
\]  

(2.6)
where \( K_w(z) = \frac{1}{1-\overline{w}z} \) is the reproducing kernel of \( H^2 \) at the point \( w \in \mathbb{D} \). We observe that

\[
H_K_w(g(z)) = \sum_{n \geq 0} \sum_{k \geq 0} \overline{w}^n k \hat{g}(k) z^n = g(\overline{w}) \sum_{n \geq 0} \overline{w}^n z^n = g(\overline{w}) K_w(z).
\]

\( H_K_w \) is a rank-one operator and as a consequence

\[
\|H_K_w\|_{S_1} = \|H_K_w\| = \frac{1}{1-|w|^2}.
\]

By the density of step-functions on \( L^1(\mathbb{D}, dA(w)) \), the Proposition 1.1.18. and (2.6)

\[
\left\| H_f \right\|_{S_1} \leq \int_{\mathbb{D}} \frac{1-|w|^2}{|w|^2} |f''(w)| \left\| H_K_w \right\|_{S_1} dA(w),
\]

\[
\left\| H_f \right\|_{S_1} \leq \int_{\mathbb{D}} \frac{|f''(w)|}{|w|^2} dA(w) < C \left\| f \right\|_{B_1} < +\infty,
\]

(2.7)

where \( C > 0 \) is an absolute constant.

Conversely, let \( H_f \in S_1 \). The canonical decomposition of \( H_f \) is

\[
H_f(g) = \sum_{n \geq 1} \langle s_n g, e_n \rangle \sigma_n.
\]

By Cauchy’s formula

\[
f''(z) = \frac{1}{\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{(1-ze^{-i\theta})^3} e^{-2i\theta} d\theta
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{e^{-i\theta}}{(1-ze^{-i\theta})^2} \frac{e^{-i\theta}}{(1-ze^{-i\theta})^2} d\theta
\]

\[
= 2 \langle H_f(g_z(w)), h_z(w) \rangle,
\]

where

\[
g_z(w) = \frac{w}{(1-zw)^2}, \quad h_z(w) = \frac{w}{(1-\overline{w}z)^2}.
\]

By Cauchy-Schwarz inequality

\[
\int_{\mathbb{D}} |f''(z)|^2 dA(z) \leq 2 \sum_{n \geq 1} s_n \int_{\mathbb{D}} |\langle g_z(w), e_n \rangle| |\langle h_z(w), \sigma_n \rangle| dA(z)
\]

\[
\leq 2 \sum_{n \geq 1} s_n \left( \int_{\mathbb{D}} |\langle g_z(w), e_n \rangle|^2 dA(z) \right)^{\frac{1}{2}} \left( \int_{\mathbb{D}} |\langle h_z(w), \sigma_n \rangle|^2 dA(z) \right)^{\frac{1}{2}}.
\]
It is sufficient to prove that there exists a constant $C > 0$ such that for every unit vector $\phi \in H^2$

$$\int_{D} |\langle g_z(w), \phi(w) \rangle|^2 dA(z), \int_{D} |\langle h_z(w), \phi(w) \rangle|^2 dA(z) \leq C < +\infty.$$ 

$$\int_{D} |\langle g_z(w), \phi(w) \rangle|^2 dA(z) = \int_{D} \left| \sum_{n \geq 0} \frac{\Gamma(n + \frac{3}{2})}{n! \Gamma(\frac{3}{2})} \phi(n + 1) z^n \right|^2 dA(z)$$

$$= \sum_{n \geq 0} \left| \frac{\Gamma(n + \frac{3}{2})}{n! \Gamma(\frac{3}{2})} \phi(n + 1) \right|^2 \frac{1}{n + 1}$$

$$= \sum_{n \geq 0} \left| \frac{(n + \frac{1}{2})(n - \frac{1}{2}) \cdots \frac{1}{\sqrt{\pi}}}{n! \frac{3}{2}} \right| |\phi(n + 1)|^2 \frac{1}{n + 1}$$

$$\leq C \sum_{n \geq 0} |\phi(n + 1)|^2 \leq C < +\infty.$$ 

Similarly,

$$\int_{D} |\langle h_z(w), \phi(w) \rangle|^2 dA(z) \leq C < +\infty.$$ 

\[\blacksquare\]

**Definition 2.4.1.** We define the operator $F : S_{\infty} \rightarrow H(D)$ as

$$F(T)(z) = \sum_{n \geq 0} t_n z^n, \ t_n = t_n(T) = \sum_{k=0}^{n} \langle T(z^k), z^{n-k} \rangle.$$ 

By the boundedness of $T$ follows that the sequence $\left\{ \frac{t_n(T)}{n+1} \right\}_{n \geq 0}$ is bounded. Thus, $F$ is well defined.

**Proposition 2.4.1.** Let $f \in \mathcal{B}_1$ and $T$ be a bounded operator on $H^2$. Then,

$$tr(T^*H_f) = \sum_{n \geq 0} \hat{f}(n) \overline{t_n}.$$ 

**Proof.** Since $f \in \mathcal{B}_1$, by the Theorem 2.4.2. and the Corollary 1.1.2. we obtain that $T^*H_f \in S_1$. By Fubini’s theorem

$$tr(T^*H_f) = \sum_{n \geq 0} \langle T^*H_f(z^n), z^n \rangle = \sum_{n \geq 0} \langle H_f(z^n), T(z^n) \rangle$$

$$= \sum_{n \geq 0} \sum_{k \geq 0} \langle H_f(z^n), z^k \rangle \overline{T(z^n), z^k} = \sum_{n \geq 0} \sum_{k \geq 0} \hat{f}(n + k) \overline{T(z^n), z^k}$$

$$= \sum_{m \geq 0} \sum_{n \geq m} \hat{f}(n) \overline{T(z^m), z^{n-m}} = \sum_{n \geq 0} \hat{f}(n) \sum_{m=0}^{n} \overline{T(z^m), z^{n-m}}$$

$$= \sum_{n \geq 0} \hat{f}(n) \overline{t_n}.$$
Proposition 2.4.2. The operator $F$ maps $S_\infty$ boundedly into $B^{-1}_\infty$.

Proof. For $T \in S_\infty$, we define the linear functional $L : \mathcal{B}_1 \to \mathbb{C}$ as

$$L(f) = \sum_{n \geq 0} \hat{f}(n)\bar{f}_n.$$ 

By the Proposition 2.4.1., the Corollary 1.1.4. and the equation (2.7)

$$|L(f)| = |\text{tr}(T^*H_f)| \leq \|T\| \|H_f\|_{S_1} \leq C \|T\| \|f\|_{\mathcal{B}_1}.$$ (2.8)

This implies that $L \in (\mathcal{B}_1)^*$ and by the Theorem 1.2.15.

$$F(T) \in B^{-1}_\infty.$$

The constant $C > 0$ in (2.8) is independent of $T \in S_\infty$ or $f \in \mathcal{B}_1$. The proof follows by the duality Theorem 1.2.15. \hfill \blacksquare

Lemma 2.4.1. Let $f$ be a continuous function on the closed interval $[0, 1]$ and

$$E_t = \{x \in [0, 1] : |f(x)|(1-x)^2 > t\}, \ t > 0.$$ 

Then, there exists a constant $C > 0$ such that

$$\int_{E_t} \frac{dx}{(1-x)^2} \leq C \left( \int_0^1 |f(x)|^2(1-x)dx \right)^{\frac{1}{2}}, \ \forall t > 0.$$ 

Proof. We observe that

$$\int_0^1 |f(x)|^2(1-x)dx \geq \int_{E_t} |f(x)|^2(1-x)dx \geq t^2 \int_{E_t} \frac{dx}{(1-x)^3}$$

and as a consequence

$$\frac{1}{t} \left( \int_0^1 |f(x)|^2(1-x)dx \right)^{\frac{1}{2}} \geq \left( \int_{E_t} \frac{dx}{(1-x)^3} \right)^{\frac{1}{2}}.$$ (2.9)

If we make the substitution $y = \frac{1}{1-x}$ in the integral in the right-hand side of (2.9) it is sufficient to prove that there exists a constant $C > 0$ such that for every open set $E \subset [1, +\infty)$

$$\int_E dx \leq C \left( \int_E xdx \right)^{\frac{1}{2}}.$$ (2.10)
Let $I_1 = (a_1, b_1), I_2 = (a_2, b_2)$ be disjoint open intervals with $1 \leq a_1, b_1 \leq a_2$. Then,

$$\int_{I_1} x \, dx \geq \int_1^{m(I_1)+1} x \, dx \quad \text{and} \quad \int_{I_2} x \, dx \geq \int_{m(I_1)+1}^{m(I_1)+m(I_2)+1} x \, dx,$$

where $m$ is the Lebesgue measure of $\mathbb{R}$.

Every open set on $\mathbb{R}$ can be written as a countable union of disjoint open intervals, this and (2.11) imply that

$$\int_{E} x \, dx \geq \int_1^{m(E)+1} x \, dx \geq \frac{m(E)^2}{2}.$$

So, (2.10) holds for $C = \sqrt{2}$ and this completes the proof.

**Theorem 2.4.3. (Littlewood-Paley)** There exists a constant $C > 0$ such that for every $f \in H^1$

$$\int_0^{2\pi} \left( \int_0^1 |f'(re^{i\theta})|^2 (1-r) \, dr \right)^{1/2} \, d\theta \leq C \|f\|_{H^1}.$$

**Proof.** For $f \equiv 0$ the proof follows, we assume that $f \not\equiv 0$. Suppose $g$ is an arbitrary function in $H^2$, by Tonelli’s theorem

$$\int_0^{2\pi} \int_0^1 |g'(re^{i\theta})|^2 (1-r) \, dr \, d\theta = 2\pi \int_0^1 (1-r) \sum_{n \geq 1} n^2 |\hat{g}(n)|^2 r^{2n-2} \, dr$$

$$= \pi \sum_{n \geq 1} \frac{n}{2n-1} |\hat{g}(n)|^2 \leq \pi \|g\|_{H^2}^2. \quad (2.12)$$

By the Hardy-Littlewood theorem [5] there exists a constant $C_1 > 0$ such that

$$\left( \int_0^{2\pi} \sup_{r \in [0,1)} |g(re^{i\theta})|^2 \, d\theta \right)^{1/2} \leq C_1 \|g\|_{H^2}, \quad \forall g \in H^2. \quad (2.13)$$
We assume first that \( f(z) \neq 0, \forall z \in \mathbb{D} \), there exists a function \( g \in H^2 \) such that \( f = g^2, \|f\|_{H^1} = \|g\|_{H^2}^2 \). By the Cauchy-Schwarz inequality, (2.12) and (2.13)

\[
\int_0^{2\pi} \left( \int_0^1 |f'(re^{i\theta})|^2 (1 - r) dr \right)^{\frac{1}{2}} d\theta = 2 \int_0^{2\pi} \left( \int_0^1 |g(re^{i\theta})|^2 |g'(re^{i\theta})|^2 (1 - r) dr \right)^{\frac{1}{2}} d\theta \\
\leq 2 \int_0^{2\pi} \sup_{r \in [0,1]} |g(re^{i\theta})|^2 \left( \int_0^1 |g'(re^{i\theta})|^2 (1 - r) dr \right)^{\frac{1}{2}} d\theta \\
\leq 2 \left( \int_0^{2\pi} \sup_{r \in [0,1]} |g(re^{i\theta})|^2 d\theta \right)^{\frac{1}{2}} \left( \int_0^1 \left( \int_0^1 |g'(re^{i\theta})|^2 (1 - r) dr d\theta \right) \right)^{\frac{1}{2}} \\
\leq C \|g\|_{H^2}^2 = C \|f\|_{H^1},
\]

where \( C > 0 \) is a constant independent of \( f \in H^1 \).

Suppose \( f \) is an arbitrary function in \( H^2 \), by the canonical factorization theorem \( f = Bh \), where \( B \) is the corresponding Blaschke product and \( h(z) \neq 0, \forall z \in \mathbb{D} \). We consider the decomposition \( f = f_1 + f_2 \), where \( f_1 = \frac{B - 1}{2} h, f_2 = \frac{B + 1}{2} h \). It is easy to prove that \( \|f_1\|_{H^1}, \|f_2\|_{H^1} \leq \|f\|_{H^1} \).

\[
\int_0^{2\pi} \left( \int_0^1 |f'(re^{i\theta})|^2 (1 - r) dr \right)^{\frac{1}{2}} d\theta \leq \int_0^{2\pi} \left( \int_0^1 |f_1'(re^{i\theta})|^2 (1 - r) dr \right)^{\frac{1}{2}} d\theta \\
+ \int_0^{2\pi} \left( \int_0^1 |f_2'(re^{i\theta})|^2 (1 - r) dr \right)^{\frac{1}{2}} d\theta \leq 2C \|f\|_{H^1}.
\]

\[\blacksquare\]

**Lemma 2.4.2.** There exists a constant \( C > 0 \) such that for every \( f \in H^1 \) and \( t > 0 \)

\[
\int_{E_t} \frac{dA(z)}{(1 - |z|^2)^2} \leq C \frac{t}{t} \|f\|_{H^1}, \text{ where } E_t = \{ z \in \mathbb{D} : (1 - |z|^2)^2 |f'(z)| > t \}.
\]

**Proof.** We consider the sets

\[E_t(\theta) = \{ r \in [0, 1) : (1 - |r|^2)^2 |f'(re^{i\theta})| > t \}.\]
CHAPTER 2. ADDITIVE HANKEL OPERATORS

By the Lemma 2.4.1 and the Theorem 2.4.3, there exist constants $C, C_1 > 0$ independent of $t > 0$ and $f \in H^1$, such that

$$\int_{E_t} \frac{dA(z)}{(1-|z|^2)^2} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\frac{1}{1-\rho^2}} r \, dr \, d\theta$$

$$\leq \frac{C_1}{t} \left( \int_0^{2\pi} \left( \int_0^1 |f'(re^{i\theta})|^2 (1-r) \, dr \right)^{\frac{1}{2}} \, d\theta \right)$$

$$\leq \frac{C}{t} \|f\|_{H^1}.$$  

**Lemma 2.4.3.** The operator $F$ maps $S_1$ boundedly into $H^1$.

**Proof.** Suppose $T \in S_1$ and the canonical decomposition

$$T(g) = \sum_{n \geq 1} s_n \langle g, e_n \rangle \sigma_n, \quad g \in H^2.$$  

We observe that

$$F(T)(z) = \sum_{n \geq 1} s_n F(T_n)(z),$$

where $T_n(g) = \langle g, e_n \rangle \sigma_n$. It is sufficient to prove that $\|F(S)\|_{H^1} \leq 1$ for every rank-one operator of the form $S(g) = \langle g, \phi \rangle \psi$, where $\|\phi\|_{H^2} = \|\psi\|_{H^2} = 1$.

$$F(S)(z) = \sum_{n \geq 0} t_n z^n, \quad t_n = \sum_{k=0}^n \langle S(z^k), z^{n-k} \rangle = \sum_{k=0}^n \hat{\psi}(n-k) \hat{\phi}(k).$$

We observe that

$$\|F(S)(z)\|_{H^1} = \left\| \frac{\phi(z)\psi(z)}{1-|z|^2} \right\|_{H^1} \leq \|\phi\|_{H^2} \|\psi\|_{H^2} = 1.$$  

This completes the proof.

**Theorem 2.4.4.** If $p \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the operator $F : S_\infty \to B_{q^{-1}}$ maps $S_p$ boundedly into $B_{p^{-\frac{1}{q}}}^{-1}$.

**Proof.** Suppose $d\mu(z) = \frac{dA(z)}{(1-|z|^2)^2}$. It is sufficient to prove that the operator

$$W(T) = (1-|z|^2)^2 |(F(T))'(z)|$$
maps $S_p$ boundedly into $L^p(\mathbb{D}, \mu)$. By the Proposition 2.4.2. $W$ maps $S_\infty$ boundedly into $L^\infty(\mathbb{D}, \mu)$. By the Lemma 2.4.2 and the Lemma 2.4.3., there exists a constant $C > 0$ such that

$$\mu(\{x : W(T)(x) > t\}) \leq \frac{C}{t} \|T\|_{S_1}, \quad \forall t > 0, \quad \forall T \in S_1.$$ 

The proof follows by the Marcinkiewicz Interpolation Theorem 1.1.14. ■

**Theorem 2.4.5. (Peller)** Let $p \in [1, +\infty)$ and $f \in H(\mathbb{D})$. Then, $H_f \in S_p$ if and only if $f \in B_p$.

**Proof.** We have already proved the Theorem for $p = 1$. For $p \in (1, +\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$, we assume that $f \in B_p$, we define the linear functional

$$L : S_q \to \mathbb{C}, \quad L(T) = \text{tr}(T^*H_f) = \sum_{n \geq 0} \hat{f}(n) t_n(T).$$

By the Theorem 2.4.4. and the Theorem 1.2.15., there exist constants $C_1, C > 0$ such that

$$|L(T)| = \left| \sum_{n \geq 0} \hat{f}(n) t_n(T) \right| \leq C_1 \|f\|_{B_p} \|F(T)\|_{\mathcal{B}^{-\frac{1}{q}}} \leq C \|f\|_{B_p} \|T\|_{S_q}.$$ 

By the Lemma 1.1.5. follows that

$$\|H_f\|_{S_p} \leq C \|f\|_{B_p} < +\infty.$$ 

Conversely, we assume that $H_f \in S_p$. By the Theorem 2.4.4. $F(H_f)(z) \in \mathcal{B}^{-\frac{1}{q}}_p$ and by the Theorem 1.2.11.

$$F(H_f)(z) \in A^p_{p-2}, \quad F(H_f)(z) = \sum_{n \geq 0} t_n z^n.$$ 

$$t_n = \sum_{k=0}^{n} \langle H_f(z^k), z^{n-k} \rangle = \sum_{k=0}^{n} \hat{f}(n) = (n+1)\hat{f}(n).$$

Follows that

$$\int_{\mathbb{D}} \left| \sum_{n \geq 0} (n+1)\hat{f}(n) z^n \right|^p (1-|z|^2)^{p-2} dA(z) = \int_{\mathbb{D}} |F(H_f)(z)|^p (1-|z|^2)^{p-2} dA(z) < +\infty.$$ 

By Minkowski's inequality $\|f\|_{B_p} < +\infty$. ■

**Theorem 2.4.6. (Peller, Semmes)** Let $p \in (0,1)$ and $f \in H(\mathbb{D})$. Then, $H_f \in S_p$ if and only if $f \in B_p$.

**Proof.** [22]
Part II

Multiplicative Hankel Operators
Chapter 3

Preliminaries

3.1 Dirichlet Series

Definition 3.1.1. A Dirichlet series is a series of the form

\[ f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}, \]

where \( s = \sigma + it \in \mathbb{C} \) and \( \{a_n\}_{n \geq 1} \) is a sequence of complex numbers.

The primary example of a Dirichlet series is the Riemann zeta function

\[ \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}. \]

We will denote by \( \mathbb{C}_a, a \in \mathbb{R} \) the half-plane

\[ \mathbb{C}_a = \{z \in \mathbb{C} : Re(z) > a\}. \]

Suppose \( \{a_n\}_{n \geq 1} \) is a sequence of complex numbers, we denote by \( A(t) \) the summatory function

\[ A(t) = \sum_{n \leq t} a_n, \quad t > 0, \quad A(t) = 0, \quad t \leq 0. \]

If in addition \( \sum_{n \geq 1} a_n \) converges, we define the function

\[ R(t) = \sum_{n > t} a_n, \quad t \in \mathbb{R}. \]

Proposition 3.1.1. (Summation by Parts) Let \( \{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 1} \) be two sequences of complex numbers and \( A(t), R(t) \) as above. Then, for every \( N, M \in \mathbb{N} \)

1.

\[ \sum_{k=M}^{N} a_k b_k = A(N)b_N - A(M-1)b_M - \sum_{k=M}^{N-1} A(k)(b_{k+1} - b_k). \]
2. If \( \sum_{n=1}^{\infty} a_n \) converges,

\[
\sum_{k=M}^{N} a_k b_k = R(M-1)b_M - R(N)b_N + \sum_{k=M}^{N-1} R(k)(b_{k+1} - b_k).
\]

Proof.

1. 

\[
\sum_{k=M}^{N} a_k b_k = \sum_{k=M}^{N} [A(k) - A(k-1)]b_k = \sum_{k=M}^{N} A(k)b_k - \sum_{k=M}^{N-1} A(k)b_{k+1}
\]

\[
= A(N)b_N - A(M-1)b_M - \sum_{k=M}^{N-1} A(k)(b_{k+1} - b_k).
\]

2. 

\[
\sum_{k=M}^{N} a_k b_k = \sum_{k=M}^{N} [R(k-1) - R(k)]b_k = \sum_{k=M-1}^{N-1} R(k)b_{k+1} - \sum_{k=M}^{N-1} R(k)b_k
\]

\[
= R(M-1)b_M - R(N)b_N + \sum_{k=M}^{N-1} R(k)(b_{k+1} - b_k).
\]

\[\blacksquare\]

Lemma 3.1.1. Let \( s = \sigma + it \in \mathbb{C}_0 \), then for every \( n \in \mathbb{N} \)

\[
\left| \frac{1}{(n+1)^s} - \frac{1}{n^s} \right| \leq \frac{|s|}{\sigma} \left( \frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right).
\]

Proof. We observe that

\[
\left| \frac{1}{(n+1)^s} - \frac{1}{n^s} \right| = \left| \int_{n}^{n+1} \frac{s}{x^{s+1}} \, dx \right| \leq \int_{n}^{n+1} \frac{|s|}{x^{\sigma+1}} \, dx = \frac{|s|}{\sigma} \left( \frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right).
\]

\[\blacksquare\]

Theorem 3.1.1. Let \( f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \) be a Dirichlet series that converges at a point \( s_0 = \sigma_0 + it_0 \). Then, \( f(s) \) converges throughout the half-plane \( \mathbb{C}_{\sigma_0} \) and converges uniformly on every angular subsector

\[
\Omega_\phi := \{ z \in \mathbb{C} : |arg(z - s_0)| \leq \phi \}, \quad \phi \in \left( 0, \frac{\pi}{2} \right).
\]
3.1. DIRICHLET SERIES

Proof. Without loss of generality we can assume that \( s_0 = 0 \). The series \( \sum_{n \geq 1} a_n \) is convergent and as a consequence for every \( \epsilon > 0 \) there exists a \( n_0 = n_0(\epsilon) \in \mathbb{N} \) such that

\[
|R(t)| \leq \epsilon, \quad \forall t \geq n_0.
\]

Suppose \( s \in \Omega_\phi \) and \( N, M > n_0 \), by the Lemma 3.1.1. and the Proposition 3.1.1.

\[
\left| \sum_{k=M}^{N} \frac{a_k}{k^s} \right| \leq \left| R(M-1) \frac{1}{M^s} \right| + \left| R(N) \frac{1}{N^s} \right| + \sum_{k=M}^{N-1} |R(k)| \left| \frac{1}{(k+1)^s} - \frac{1}{k^s} \right| \\
\leq 2\epsilon + \frac{|s|}{\sigma} \sum_{k=M}^{N-1} \left( \frac{1}{k^s} - \frac{1}{(k+1)^s} \right) \leq 2\epsilon + 2\epsilon \frac{|s|}{\sigma} \leq 2\epsilon + 2\epsilon \frac{1}{\cos(\phi)}.
\]

This implies that the sequence of the partial sums \( \left\{ \sum_{k=1}^{n} \frac{a_k}{k^s} \right\}_{n \geq 1} \) is uniformly Cauchy on \( \Omega_\phi \). So, \( f(s) \) converges uniformly on every angular subsector and pointwise on \( C_0 \), since \( C_0 = \bigcup_{\phi \in (0, \frac{\pi}{2})} \Omega_\phi \). □

Corollary 3.1.1. Let \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \) be a Dirichlet series that converges absolutely at a point \( s_0 = \sigma_0 + it_0 \). Then, \( f(s) \) converges absolutely throughout the half-plane \( C_{\sigma_0} \).

Proof. The proof follows if we apply the Theorem 3.1.1. on the Dirichlet series \( \sum_{n \geq 1} \frac{|a_n|}{n^s} \). □

Definition 3.1.2. For a Dirichlet series \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \), the abscissas of convergence and absolute convergence are defined respectively by

1. \( \sigma_c = \inf\{\sigma \in \mathbb{R} : f(\sigma) \text{ converges}\} \).

2. \( \sigma_a = \inf\{\sigma \in \mathbb{R} : f(\sigma) \text{ converges absolutely}\} \).

If \( f(s) \) is nowhere convergent (absolutely convergent), then \( \sigma_c = +\infty (\sigma_a = +\infty) \). If \( f(s) \) is everywhere convergent (absolutely convergent), then \( \sigma_c = -\infty (\sigma_a = -\infty) \).

Theorem 3.1.2. Let \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \) be a Dirichlet series. Then,

\[
\sigma_c \leq \sigma_a \leq \sigma_c + 1.
\]

Proof. Trivially \( \sigma_c \leq \sigma_a \). For \( \epsilon > 0 \), we observe that there exists a constant \( C > 0 \) such that

\[
\frac{|a_n|}{n^{\sigma_c + \epsilon}} < C.
\]
Thus, the series converges absolutely at the point $s = \sigma_c + 1 + 2\varepsilon$

$$\sum_{n \geq 1} \frac{|a_n|}{n^s} = \sum_{n \geq 1} \frac{|a_n|}{n^{\sigma_c + 1 + \varepsilon}} \leq C \sum_{n \geq 1} \frac{1}{n^{1 + \varepsilon}} < +\infty.$$ 

It follows that

$$\sigma_c \leq \sigma_a \leq \sigma_c + 1.$$  

The above inequality is optimal. We consider the alternating zeta function

$$f(s) = \sum_{n \geq 1} \frac{(-1)^n}{n^s}.$$ 

We observe that $f(\sigma)$ converges for $\sigma > 0$ by the Leibniz test and diverges for $\sigma \leq 0$. This implies that $\sigma_c = 0$ and trivially $\sigma_a = 1$.

**Theorem 3.1.3.** Suppose $f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$ is a Dirichlet series.

1. If $\sum_{n \geq 1} a_n$ converges, then

$$\sigma_c = \limsup_{n \to \infty} \frac{\log |R(n)|}{\log n} \leq 0.$$ 

2. If $\sum_{n \geq 1} a_n$ diverges, then

$$\sigma_c = \limsup_{n \to \infty} \frac{\log |A(n)|}{\log n} \geq 0.$$ 

**Proof.**

1. First we will prove that

$$\limsup_{n \to \infty} \frac{\log |R(n)|}{\log n} \leq \sigma_c.$$  \hspace{1cm} (3.1)

If $\sigma_c = 0$, then (3.1) holds since $\log |R(n)| < 0$ for $n$ sufficiently large.

If $\sigma_c < 0$, we consider an arbitrary point $\sigma < 0$ such that $f(\sigma)$ converges and we define

$$b_n = \frac{a_n}{n^\sigma}, \quad B(t) = \sum_{n \leq t} b_n.$$ 

There exists a constant $K < +\infty$ such that

$$|B(t)| < K, \quad \forall t \geq 0.$$
We observe that

\[ R(N) = \sum_{n>N} b_n n^\sigma = \sum_{n>N} B(n)(n^\sigma - (n+1)^\sigma) - B(N)(N+1)^\sigma, \]

\[ |R(N)| \leq K \sum_{n>N} \left( n^\sigma - (n+1)^\sigma \right) + K(N+1)^\sigma \leq 2KN^\sigma. \]

Thus,

\[ \frac{\log |R(n)|}{\log n} \leq \frac{\log(2K)}{\log n} + \sigma, \]

\[ \limsup_{n \to \infty} \frac{\log |R(n)|}{\log n} \leq \sigma. \]

(3.1) follows from this.

It remains to prove that

\[ \limsup_{n \to \infty} \frac{\log |R(n)|}{\log n} \geq \sigma_c. \] (3.2)

If \( \limsup_{n \to \infty} \frac{\log |R(n)|}{\log n} = 0 \), then (3.2) holds since \( \sigma_c \leq 0 \).

If \( \limsup_{n \to \infty} \frac{\log |R(n)|}{\log n} < 0 \), suppose \( \epsilon > 0 \) is sufficiently small and

\[ 0 > \sigma = 2\epsilon + \limsup_{n \to \infty} \frac{\log |R(n)|}{\log n}. \]

We observe that there exists a \( n_0 \in \mathbb{N} \) such that for every \( n \geq n_0 \)

\[ \frac{\log |R(n)|}{\log(n)} \leq \sigma - \epsilon. \]

\[ |R(n)| \leq n^{\sigma-\epsilon}, \ \forall n > n_0. \] (3.3)

For \( M, N \in \mathbb{N} \) such that \( N > M > n_0 \)

\[ \sum_{n=M}^{N} \frac{a_n}{n^\sigma} = R(M-1) - \frac{1}{M^\sigma} - R(N) + \sum_{n=M}^{N-1} R(n) \left( \frac{1}{(n+1)^\sigma} - \frac{1}{n^\sigma} \right). \]

By (3.3) there exists a constant \( C_1 > 0 \) such that

\[ \left| \sum_{n=M}^{N} \frac{a_n}{n^\sigma} \right| \leq C_1 M^{-\epsilon}. \]

The sequence of partial sums \( \left\{ \sum_{k=1}^{n} \frac{a_k}{k^\sigma} \right\}_{n \geq 1} \) is a Cauchy sequence and as a consequence convergent, (3.2) follows from this. So,

\[ 0 \geq \sigma_c = \limsup_{n \to \infty} \frac{\log |R(n)|}{\log n}. \]
2. The proof is similar to 1.

Lemma 3.1.2. Suppose \( a, T > 0 \).

1. For \( x \in (1, +\infty) \)

\[
\left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{x^z}{z} \, dz \right| \leq \frac{x^a}{T|\log x|\pi}.
\]

2. For \( x \in (0, 1) \)

\[
\left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{x^z}{z} \, dz \right| \leq \frac{x^a}{T|\log x|\pi}.
\]

Proof.

1. Suppose \( \gamma > 0 \) is sufficiently large and \( R_\gamma \) is the positively oriented rectangle with vertices at \( a \pm iT \) and \( a - \gamma \pm iT \).

\[\text{Figure 3.1:}\]

By Cauchy’s formula

\[
\frac{1}{2\pi i} \int_{R_\gamma} \frac{x^z}{z} \, dz = 1.
\] (3.4)
3.1. DIRICHLET SERIES

We observe that

\[
\left| \int_{\mathbb{R}^1} \frac{x^z}{z} \, dz \right| = \left| \int_{-T}^{T} \frac{x^{a-\gamma+it}}{a-\gamma+it} \, dt \right| \leq T \int_{-T}^{T} \frac{x^{a-\gamma}}{\gamma-a} \, dt = 2T \frac{x^{a-\gamma}}{\gamma-a}.
\]

\[
\left| \int_{\mathbb{R}_2,\mathbb{R}_4} \frac{x^z}{z} \, dz \right| = \left| \int_{a-\gamma}^{a} \frac{x^{t+iT}t}{t+iT} \, dt \right| \leq \int_{a-\gamma}^{a} \frac{x^{t}}{T} \, dt \leq \frac{x^{a}}{T \log x}.
\]

Letting \( \gamma \to +\infty \) follows that

\[
\left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{x^z}{z} \, dz - 1 \right| \leq \frac{x^{a}}{T |\log x| \pi}.
\]

2. Suppose \( \gamma > 0 \) is sufficiently large and \( R_\gamma \) is the positively oriented rectangle with vertices at \( a \pm iT \) and \( a + \gamma \pm iT \).

![](image)

**Figure 3.2:**

By Cauchy’s formula

\[
\frac{1}{2\pi i} \int_{R_\gamma} \frac{x^z}{z} \, dz = 0.
\]
We observe that
\[ \int_{\mathbb{R}^3} x^z z \, dz = \int_{-T}^{T} \frac{x^{a+y+it}}{a+y+it} \, dt \leq \int_{-T}^{T} \frac{x^{a+y}}{a+y} \, dt = 2T \frac{x^{a+y}}{a+y}. \]
\[ \int_{\mathbb{R}_{2,R4}} x^z z \, dz = \int_{a}^{a+\gamma} \frac{x^{t+iT}}{t+iT} \, dt \leq \int_{a}^{a+\gamma} \frac{x^{t}}{T} \, dt \leq \frac{x^a}{T |\log x|}. \]

Letting \( \gamma \to +\infty \) follows that
\[ \left| \frac{1}{a-iT} \int_{a-iT}^{a+iT} f(z) \frac{x^z}{z} \, dz \right| \leq \frac{x^a}{T |\log x|}. \]

**Theorem 3.1.4. (Perron’s Formula)** Let \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \) be a Dirichlet series and \( +\infty > a > \max(0, \sigma_n) \). Then, for every \( T > 0 \) and \( x \geq 1 \) that is not an integer
\[ \left| A(x) - \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f(z) \frac{x^z}{z} \, dz \right| \leq \frac{x^a}{\pi T} \sum_{n \geq 1} |a_n| |n^a| |\log(\frac{z}{n})|. \]

**Proof.** We observe that \( f(s) \) converges absolutely on \([a-iT, a+iT]\), so we can interchange the order of the integration
\[ \left| A(x) - \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f(z) \frac{x^z}{z} \, dz \right| = \sum_{n \geq 1} a_n \left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{\left(\frac{z}{n}\right)^x}{z} \, dz \right| - A(x) \]
\[ \leq \sum_{n \leq x} |a_n| \left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{\left(\frac{z}{n}\right)^x}{z} \, dz \right| \]
\[ + \sum_{n > x} |a_n| \left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{\left(\frac{z}{n}\right)^x}{z} \, dz \right|. \]

By the Lemma 3.1.2.
\[ \left| A(x) - \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f(z) \frac{x^z}{z} \, dz \right| \leq \frac{x^a}{\pi T} \sum_{n \geq 1} |a_n| |n^a| |\log(\frac{z}{n})|. \]
3.1. DIRICHLET SERIES

Corollary 3.1.2. Let \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \) be a Dirichlet series, \(+\infty > a \geq \max(0, \sigma_a)\) and \( x \geq 1 \) that is not an integer. Then,

\[
A(x) = \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f(z) \frac{x^z}{z} \, dz.
\]

Definition 3.1.3. For a Dirichlet series \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \), the abscissas of uniform and bounded convergence are defined respectively by

1. \( \sigma_u = \inf\{\sigma \in \mathbb{R} : f(s) \text{ converges uniformly on the half-plane } C_{\sigma}\} \).
2. \( \sigma_b = \inf\{\sigma \in \mathbb{R} : f(s) \text{ converges and it is bounded on the half-plane } C_{\sigma}\} \).

Proposition 3.1.2. For every Dirichlet series \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \),

\[
\sigma_c \leq \sigma_b \leq \sigma_u \leq \sigma_a \leq \sigma_c + 1.
\]

Proof. By the Theorem 3.1.2. it is sufficient to prove that

\[
\sigma_b \leq \sigma_u \leq \sigma_a.
\]

For every \( \epsilon > 0 \), the Dirichlet series is the uniform limit of the corresponding sequence of partial sums on \( C_{\sigma_a+\epsilon} \) and as a consequence is bounded on \( C_{\sigma_u+\epsilon} \). Follows that \( \sigma_b \leq \sigma_u \).

For \( \delta, \epsilon > 0 \), there exists a \( n_1 \in \mathbb{N} \) such that for every \( N \geq n_1 \)

\[
\sum_{n \geq N} \frac{|a_n|}{n^{\sigma_a+\delta}} \leq \epsilon,
\]

\[
\left| f(s) - \sum_{n=1}^{N} \frac{a_n}{n^s} \right| \leq \sum_{n>N} \frac{|a_n|}{n^{\sigma_a+\delta}} \leq \sum_{n>N} \frac{|a_n|}{n^{\sigma_a}} \leq \epsilon, \quad \forall s = \sigma + it \in C_{\sigma_a+\delta}.
\]

Since \( \delta, \epsilon > 0 \) were arbitrarias follows that

\[
\sigma_u \leq \sigma_a.
\]

\[\blacksquare\]

Proposition 3.1.3. Let \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \) be a Dirichlet series that diverges at 0. Then,

\[
\sigma_u = \limsup_{N \to +\infty} \log \left\| \sum_{n=1}^{N} \frac{a_n}{n^s} \right\|_\infty / \log N.
\]
Proof. Suppose
\[ \gamma := \limsup_{N \to +\infty} \frac{\log \left\| \sum_{n=1}^{N} \frac{a_n}{n^\sigma} \right\|_\infty}{\log N}. \]
We will first prove that
\[ \sigma_u \leq \gamma. \]  \hspace{1cm} (3.6)
Without loss of generality we can assume that \( \gamma < +\infty \). We consider the functions
\[ g_N(t) = \sum_{n=1}^{N} \frac{a_n}{n^\sigma}, \quad N \in \mathbb{N}, \quad g_0 \equiv 0. \]
For every \( \epsilon > 0 \) there exists a \( n_0(\epsilon) \in \mathbb{N} \) such that
\[ \|g_N\|_\infty \leq N^\gamma + \epsilon, \quad \forall N > n_0. \]
For \( s = \sigma + it \in \mathbb{C}_{\gamma+2\epsilon} \) and \( M, N \in \mathbb{N} \) such that \( N > M > n_0 \)
\[ \left| \sum_{n=M}^{N} \frac{a_n}{n^s} \right| = \left| \sum_{n=M}^{N} \frac{g_n(t) - g_{n-1}(t)}{n^\sigma} \right| \leq \frac{g_N(t)}{N^\sigma} + \frac{g_M(t)}{M^\sigma} + \sum_{n=M}^{N-1} |g_n(t)| \left( \frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right) \leq N^{\gamma+\epsilon-\sigma} + 2M^{\gamma+\epsilon-\sigma} \leq \frac{3}{M^\epsilon}. \]
This implies that \( f(s) \) converges uniformly on \( \mathbb{C}_{\gamma+2\epsilon} \) and as a consequence \( \sigma_u \leq \gamma \).
It remains to prove that
\[ \gamma \leq \sigma_u. \]  \hspace{1cm} (3.7)
Without loss of generality we assume that \( \sigma_u < +\infty \). For \( \epsilon > 0 \), \( s = \sigma + it = \sigma_u + \epsilon + it \) and
\[ x_n = \frac{a_n}{n^s}, \quad S_N = \sum_{n=1}^{N} x_n, \quad S = \sum_{n \geq 1} x_n. \]
We observe that for \( M, N \in \mathbb{N}, \quad N > M \)
\[ \frac{1}{N^\sigma} \sum_{n=1}^{N} x_n n^\sigma = S_N - \frac{1}{N^\sigma} \sum_{n=1}^{N-1} S_n \left[ (n+1)^\sigma - n^\sigma \right] = S_N - \frac{1}{N^\sigma} \sum_{n=1}^{M-1} S_n \left[ (n+1)^\sigma - n^\sigma \right] - \frac{1}{N^\sigma} \sum_{n=M}^{N-1} S_n \left[ (n+1)^\sigma - n^\sigma \right] = S_N - \frac{1}{N^\sigma} \sum_{n=1}^{M-1} S_n \left[ (n+1)^\sigma - n^\sigma \right] - S \frac{N^\sigma - M^\sigma}{N^\sigma} - \frac{1}{N^\sigma} \sum_{n=M}^{N-1} (S_n - S) \left[ (n+1)^\sigma - n^\sigma \right] = S \frac{N^\sigma - M^\sigma}{N^\sigma} - \frac{1}{N^\sigma} \sum_{n=M}^{N-1} (S_n - S) \left[ (n+1)^\sigma - n^\sigma \right]. \]
Letting $N \to +\infty$ follows that

$$
\lim_{N \to +\infty} \left\| \frac{1}{N^\sigma} \sum_{n=1}^{N} x_n n^\sigma \right\|_\infty = 0.
$$

Thus,

$$
\limsup_{N \to +\infty} \frac{\log \left\| \sum_{n=1}^{N} \frac{a_n}{n^\sigma} \right\|_\infty}{\log N} \leq \sigma_u + \epsilon,
\gamma \leq \sigma_u.
$$

\[\square\]

**Theorem 3.1.5.** Let $f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$ be a Dirichlet series. Then,

$$
\sigma_b = \sigma_u.
$$

**Proof.** Without loss of generality we can assume that $\sigma_b < \sigma_a$. It is sufficient to prove that for every $p \in (\sigma_b, \sigma_a)$ and $\delta > 0$ sufficiently small the Dirichlet series $f(s)$ converges uniformly on $\mathbb{C}_{p+\delta}$. There exists a constant $K > 0$ such that

$$
|f(s)| \leq K, \forall s \in \mathbb{C}_p.
$$

For fixed $s = \sigma + it \in \mathbb{C}_{p+\delta}$ and $N \in \mathbb{N}$, we consider the meromorphic function

$$
g(z) = \frac{f(z)}{z-s} \left( N + \frac{1}{2} \right)^{z-s}.
$$

Suppose $R$ is the positively oriented rectangle with vertices at the points $s - \delta \pm i N^d$ and $s + (\sigma_a - p) \pm i N^d$, where $d = \sigma_a - p + 2$.

![Figure 3.3:](image-url)
By Cauchy’s formula
\[ \int_R g(z)dz = 2\pi i f(s). \] (3.8)

We shall be using the convention that \( C(K, \delta) \) will denote a positive constant, that depends on \( K, \delta \), not necessarily the same at each occurrence. We observe that
\[
\left| \int_{R_1} g(z)dz \right| \leq \int_{R_1} |g(z)||dz|
\leq \int_{-N^d}^{N^d} \frac{K}{\sqrt{\delta^2 + y^2}} \left( N + \frac{1}{2} \right)^{-\delta} dy
\leq KN^{-\delta} \left\{ \log \left[ y + \sqrt{\delta^2 + y^2} \right] \right\}_{-N^d}^{N^d}
\leq C(K, \delta)N^{-\delta}\log N. \] (3.9)

By (3.8), (3.9) and (3.10)
\[
\left| \int_{R_2, R_3} g(z)dz \right| \leq \int_{R_2, R_3} |g(z)||dz|
\leq K \int_{\sigma - \delta}^{\sigma + d - 2} \frac{(N + \frac{1}{2})^{y-\sigma}}{\sqrt{y^2 + (t \pm N^d)^2}} dy
\leq KN^{-d} \int_{\sigma - \delta}^{\sigma + d - 2} \left( N + \frac{1}{2} \right)^{y-\sigma} dy
\leq \frac{C(K, \delta)}{N^2\log N}. \] (3.10)

By (3.8), (3.9) and (3.10)
\[
\left| f(s) - \frac{1}{2\pi i} \int_{R_3} g(z)dz \right| \leq C(K, \delta)\frac{\log N}{N^\delta}. \] (3.11)

The Dirichlet series \( f(z) \) converges absolutely on \( R_3 \), so we can interchange the order of the integration
\[
\frac{1}{2\pi i} \int_{R_3} g(z)dz = \sum_{n \geq 1} \frac{a_n}{n^s} \int_{R_3} \left( \frac{N + \frac{1}{2}}{n} \right)^{-s} \frac{1}{z-s} dz \] (3.12)
3.1. DIRICHLET SERIES

By the Lemma 3.1.2. for $n > N$

$$\left| \frac{1}{2\pi i} \int_{R_3} \left( \frac{N + \frac{1}{2}}{n} \right)^{z-s} \frac{1}{z-s} \, dz \right| \leq \left( \frac{N + \frac{1}{2}}{n} \right)^{\sigma_a - p} \frac{1}{N^d \pi \left| \log \left( \frac{N + \frac{1}{2}}{n} \right) \right|}.$$ 

We observe that for $n > N$

$$\left| \log \left( \frac{N + \frac{1}{2}}{n} \right) \right| \geq - \log \left( \frac{N + \frac{1}{2}}{N + 1} \right) \geq \frac{1}{2N + 2}.$$ 

So, there exists a constant $C > 0$ such that

$$\left| \frac{1}{2\pi i} \int_{R_3} \left( \frac{N + \frac{1}{2}}{n} \right)^{z-s} \frac{1}{z-s} \, dz \right| \leq C \frac{1}{N n^{\sigma_a - p}}.$$ 

Similarly for $n \leq N$

$$\left| \frac{1}{2\pi i} \int_{R_3} \left( \frac{N + \frac{1}{2}}{n} \right)^{z-s} \frac{1}{z-s} \, dz - 1 \right| \leq \left( \frac{N + \frac{1}{2}}{n} \right)^{\sigma_a - p} \frac{1}{N^d \pi \left| \log \left( \frac{N + \frac{1}{2}}{n} \right) \right|} \leq \left( \frac{N + \frac{1}{2}}{n} \right)^{\sigma_a - p} \frac{1}{N^d \pi \log \left( \frac{N + \frac{1}{2}}{N} \right)} \leq C \frac{1}{N n^{\sigma_a - p}}.$$ 

Thus,

$$\left| f(s) - \sum_{n=1}^{N} \frac{a_n}{n^s} \right| \leq C(K, \delta) \frac{\log N}{N^\delta} + \sum_{n \leq N} \frac{|a_n|}{n^\sigma} \left| \frac{1}{2\pi i} \int_{R_3} \left( \frac{N + \frac{1}{2}}{n} \right)^{z-s} \frac{1}{z-s} \, dz - 1 \right|$$

$$+ \sum_{n > N} \frac{|a_n|}{n^\sigma} \left| \frac{1}{2\pi i} \int_{R_3} \left( \frac{N + \frac{1}{2}}{n} \right)^{z-s} \frac{1}{z-s} \, dz \right| \leq C(K, \delta) \left[ \frac{\log N}{N^\delta} + \frac{1}{N} \sum_{n \geq 1} |a_n| n^{\sigma_a + (\sigma - p)} \right] \leq C(K, \delta) \frac{\log N}{N^\delta}.$$
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This implies that \( f(s) \) converges uniformly on \( \mathbb{C}_{p+\delta} \). \( p \in (\sigma_b, \sigma_a) \) and \( \delta > 0 \) were arbitraries, so
\[
\sigma_u \leq \sigma_b.
\]

Now we are going to prove the analogues of Parseval's theorem and formula for Dirichlet series.

**Theorem 3.1.6.** Let \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \) and \( g(s) = \sum_{n \geq 1} \frac{b_n}{n^s} \) be two Dirichlet series with abscissas of uniform convergence \( \sigma_1 \) and \( \sigma_2 \) respectively. Then, for \( x > \sigma_1 \) and \( y > \sigma_2 \)
\[
\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(x+it)\overline{g(y+it)} \, dt = \sum_{n \geq 1} \frac{a_n b_n}{n^{x+y}}.
\]

**Proof.** We observe that
\[
f(x+it)\overline{g(y+it)} = \sum_{n,m \geq 1} \frac{a_n b_m}{n^x m^y}.
\]
\[
\frac{1}{2T} \int_{-T}^{T} \left( \frac{m}{n} \right)^{it} \, dt = \frac{T \log \frac{m}{n}}{T \log \frac{m}{n}}, \quad \forall m \neq n.
\]

We can interchange the order of integration
\[
\frac{1}{2T} \int_{-T}^{T} f(x+it)\overline{g(y+it)} \, dt = \sum_{n \geq 1} \frac{a_n b_n}{n^{x+y}} + \sum_{n \neq m} \frac{a_n b_n}{n^x m^y} \frac{\sin(T \log \frac{m}{n})}{T \log \frac{m}{n}}.
\]

By the dominated convergence theorem
\[
\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(x+it)\overline{g(y+it)} \, dt = \sum_{n \geq 1} \frac{a_n b_n}{n^{x+y}}.
\]

**Corollary 3.1.3.** Let \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \) be a Dirichlet series. Then, for \( \sigma > \sigma_u \)
\[
\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^2 \, dt = \sum_{n \geq 1} \frac{|a_n|^2}{n^{2\sigma}}.
\]
Corollary 3.1.4. Let \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \) be a Dirichlet series. Then, for \( \sigma > \sigma_u \)

\[
a_n = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(\sigma + it)n^{\sigma+it} dt.
\]

Proof.

\[
n^{\sigma+it} = n^{2\sigma} \frac{n^{-it}}{n^{\sigma+it}}.
\]

The proof follows from this. □

Proposition 3.1.4. For every Dirichlet series \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \)

\[
\sigma_a \leq \sigma_u + \frac{1}{2}.
\]

Proof. Without loss of generality we assume that \( \sigma_u < +\infty \). For \( \epsilon > 0 \), by the Theorem 3.1.5 and the Corollary 3.1.3. there exists a constant \( K(\epsilon) > 0 \) such that

\[
\left( \sum_{n=1}^{N} \frac{|a_n|^2}{n^{2(\sigma_u + \epsilon)}} \right)^{\frac{1}{2}} = \left( \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma_u + \epsilon + it)|^2 dt \right)^{\frac{1}{2}} \leq K < +\infty.
\]

By the Cauchy-Schwarz inequality

\[
\sum_{n=M}^{N} \frac{|a_n|}{n^{\sigma_u + \frac{1}{2} + 2\epsilon}} \leq \left( \sum_{n=M}^{N} \frac{|a_n|^2}{n^{2(\sigma_u + \epsilon)}} \right)^{\frac{1}{2}} \left( \sum_{n=M}^{N} \frac{1}{n^{1 + 2\epsilon}} \right)^{\frac{1}{2}} \leq K \left( \sum_{n=M}^{N} \frac{1}{n^{1 + 2\epsilon}} \right)^{\frac{1}{2}}.
\]

This implies that \( \left\{ \sum_{k=1}^{n} \frac{|a_k|}{k^{\sigma_u + \frac{1}{2} + 2\epsilon}} \right\}_{n \geq 1} \) is a Cauchy sequence and \( \sigma_a \leq \sigma_u + \frac{1}{2} \). □

H. F. Bohnenblust [1] proved that the above inequality is optimal.

### 3.2 Hardy Spaces of Dirichlet Series

Definition 3.2.1. We define the spaces of functions

1. \( \mathcal{D} = \left\{ f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} : \sigma_c(f) < +\infty \right\} \).

2. \( H^\infty(C_0) = \left\{ f \in H(C_0) : \|f\|_{H^\infty(C_0)} := \sup_{C_0} |f(z)| < +\infty \right\} \).

3. The Hardy space of bounded Dirichlet series is defined by \( \mathcal{H}_\infty = \mathcal{D} \cap H^\infty(C_0) \) and is equipped with the supremum norm.

Proposition 3.2.1. Let \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \in \mathcal{H}_\infty \). Then, \( |a_n| \leq \|f\|_{\mathcal{H}_\infty} , \forall n \in \mathbb{N} \).
Proof. For $\sigma > \sigma_u$ and $T, \epsilon > 0$, we consider the rectangle $R_T$ with vertices at $\epsilon \pm iT, \sigma \pm iT$.

\[ \left| \frac{1}{2T} \int_{R_1} f(z)n^z \, dz \right| \leq \| f \|_{H^\infty} n^\epsilon, \]
\[ \left| \frac{1}{2T} \int_{R_2} f(z)n^z \, dz \right| \leq \| f \|_{H^\infty} n^\sigma \frac{n^\sigma}{2T \log n}. \]

By Cauchy’s theorem and the Corollary 3.1.4. letting $T \rightarrow +\infty$ and then $\epsilon \rightarrow 0$, we obtain that

\[ |a_n| = \left| \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^T f(\sigma + it)n^{\sigma + it} \, dt \right| \leq \| f \|_{H^\infty}. \]

\[ \Box \]

Corollary 3.2.1. Let $f \in H^\infty$, then $\sigma_a \leq 1$.

Theorem 3.2.1. $H^\infty$ is a Banach space.
3.2. HARDY SPACES OF DIRICHLET SERIES

**Proof.** It is sufficient to prove that $\mathcal{H}^\infty$ is closed in $H^\infty(C_0)$. Suppose a sequence

$$\{f_m\}_{m \geq 1} \subset \mathcal{H}^\infty, \quad f_m(s) = \sum_{n \geq 1} \frac{a_m^n}{n^s} \to f \in H^\infty(C_0).$$

By the Proposition 3.2.1.

$$\left|a_m^n - a_{m_2}^n\right| \leq \|f_m - f_{m_2}\|_{\mathcal{H}^\infty}, \quad \forall n, m_1, m_2 \in \mathbb{N}.$$ 

Follows that $\{\{a_m^n\}_{n \geq 1}\}_{m \geq 1}$ converges to a sequence $\{a_n\}_{n \geq 1}$ in $\ell_\infty$. By the dominated convergence theorem, for every $s \in \mathbb{C}$

$$f(s) = \lim_{m \to +\infty} f_m(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \in \mathcal{H}^\infty.$$ 

This completes the proof. ■

**Proposition 3.2.2.** Let $f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \in \mathcal{H}^\infty$ and $S_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$, $N \geq 2$. Then, there exists an absolute constant $C > 0$ such that

$$\|S_N\|_{\mathcal{H}^\infty} \leq C \log N \|f\|_{\mathcal{H}^\infty}, \quad N \geq 2.$$ 

**Proof.** By Perron’s formula

$$\left| A(x) - \frac{1}{2\pi i} \int_{2-iT}^{2+iT} f(z) \frac{x^z}{z} \, dz \right| \leq \frac{x^2}{\pi T} \sum_{n \geq 1} \frac{|a_n|}{n^2 \log \frac{2}{n}}.$$ 

For $x = N + \frac{1}{2}$ and $T = x^3$,

$$\left| \log \frac{x}{n} \right| \geq \frac{1}{4(N + \frac{1}{2})}, \quad n \geq 1 \quad \text{and}$$

$$\left| A(x) - \frac{1}{2\pi i} \int_{2-iT}^{2+iT} f(z) \frac{x^z}{z} \, dz \right| \leq \frac{\pi}{24} \|f\|_{\mathcal{H}^\infty}. \quad (3.13)$$

We consider the rectangle $R_\delta$ with vertices at $\delta \pm iT$ and $2 \pm iT$, where $\delta = \frac{1}{\log x} \in (0, 2)$. 

---
We observe that
\[
\left| \int_{R_1} f(z) \frac{x^z}{z} \, dz \right| \leq \| f \|_{\mathcal{H}^\infty} x^\delta \int_{-T}^T \frac{dy}{\sqrt{\delta^2 + y^2}} \leq C_0 \| f \|_{\mathcal{H}^\infty} \log x,
\]
\[
\left| \int_{R_2,R_4} f(z) \frac{x^z}{z} \, dz \right| \leq \| f \|_{\mathcal{H}^\infty} \frac{x^2}{T \log x}.
\]

By Cauchy’s formula and (3.13) there exists a constant $C > 0$ such that
\[
\left| \sum_{n=1}^N a_n \right| \leq C \log N \| f \|_{\mathcal{H}^\infty}, \quad N \geq 2.
\]

Let $s_0 \in \mathbb{C}_0$, then
\[
\left| \sum_{n=1}^N \frac{a_n}{n^{s_0}} \right| \leq C \log N \| f(s + s_0) \|_{\mathcal{H}^\infty} \leq C \log N \| f \|_{\mathcal{H}^\infty}, \quad N \geq 2,
\]
\[
\| S_N \|_{\mathcal{H}^\infty} \leq C \log N \| f \|_{\mathcal{H}^\infty}, \quad N \geq 2.
\]

**Corollary 3.2.2.** Let $f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \in \mathcal{H}^\infty$, then $\sigma_u \leq 0$.

**Proof.** It is sufficient to prove that for every $\epsilon > 0$
\[
\sum_{n \geq 1} \frac{a_n}{n^\epsilon} < +\infty.
\]
By the Proposition 3.2.2. there exists a constant $C = C(f) > 0$ such that for every $M, N \in \mathbb{N}$, $N > M > 1$
\[ \left| \sum_{n=M}^{N} \frac{a_n}{n^\epsilon} \right| \leq C \frac{\log N}{N^\epsilon} + C \frac{\log(M-1)}{M^\epsilon} + C \frac{\log(N-1)}{M^\epsilon}. \]

This implies that $\left\{ \sum_{n=1}^{N} \frac{a_n}{n^\epsilon} \right\}_{N \geq 1}$ is a Cauchy sequence, the proof follows. ■

Now we are going to prove the analogue of Montel’s theorem for Dirichlet series.

**Theorem 3.2.2.** For every bounded sequence $\{f_m\}_{m \geq 1} \subset \mathcal{H}^\infty$ there exists a subsequence that converges uniformly on every half-plane $\mathbb{C}_\delta$, $\delta > 0$ to a function $f \in \mathcal{H}^\infty$.

**Proof.** Suppose $f_m(s) = \sum_{n \geq 1} \frac{a_n^m}{n^s}$. By the Proposition 3.2.1. $\{a_n^m\}_{n \geq 1}$ is bounded in $\ell_\infty$. By a standard diagonal argument there exists a subsequence such that $\lim_{k \to \infty} a_n^m = a_n$ and by Montel’s theorem we can assume that $\{f_{m_k}\}_{k \geq 1}$ converges locally uniformly to a function $f \in H^\infty(\mathbb{C}_0)$. For $s \in \mathbb{C}_1$, by the dominated convergence theorem
\[ f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \in \mathcal{H}^\infty. \]

It remains to prove that the convergence is uniformly on the arbitrary half-plane $\mathbb{C}_\delta$, $\delta > 0$. Suppose $Q_n^k(s) = \sum_{j=1}^{n} \frac{a_j^m - a_j}{j^s}$, by the Proposition 3.2.2.
\[
|f_{m_k}(s + \delta) - f(s + \delta)| = \left| \sum_{n \geq 1} \frac{Q_n^k(s) - Q_{n-1}^k(s)}{n^\delta} \right|
\leq \sum_{n=1}^{N} |a_n^m - a_n| + C_0 \|f - f_{m_k}\|_{\mathcal{H}^\infty} \sum_{n>N} \log(n) \left( \frac{1}{n^\delta} - \frac{1}{(n+1)^\delta} \right)
\leq \sum_{n=1}^{N} |a_n^m - a_n| + C(\delta) \sum_{n>N} \frac{1}{n^{\frac{3}{2} + 1}}.
\]

\[
\limsup_{k \to \infty} \|f_{m_k}(s + \delta) - f(s + \delta)\|_{\mathcal{H}^\infty} \leq C(\delta) \sum_{n>N} \frac{1}{n^{\frac{3}{2} + 1}}, \quad \forall N \in \mathbb{N},
\]
\[
\lim_{k \to \infty} \|f_{m_k}(s + \delta) - f(s + \delta)\|_{\mathcal{H}^\infty} = 0.
\]

■
Definition 3.2.2. The Hardy space $\mathcal{H}^2$ of Dirichlet series is defined by

$$\mathcal{H}^2 = \left\{ f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} : \|f\|_{\mathcal{H}^2} := \sqrt{\sum_{n \geq 1} |a_n|^2} < +\infty \right\}.$$ 

$\mathcal{H}^2$ is a separable Hilbert space equipped with the inner product

$$\langle f, g \rangle = \sum_{n \geq 1} a_n \overline{b_n}, \quad f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}, \quad g(s) = \sum_{n \geq 1} \frac{b_n}{n^s}.$$ 

The standard orthonormal basis of $\mathcal{H}^2$ is

$$\left\{ e_n(s) = \frac{1}{n^s} : n \in \mathbb{N} \right\}.$$ 

Proposition 3.2.3. Let $f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \in \mathcal{H}^2$. Then, $\sigma_a \leq \frac{1}{2}$.

Proof. By the Cauchy-Schwarz inequality for $\sigma > \frac{1}{2}$

$$\sum_{n \geq 1} \frac{|a_n|}{n^\sigma} \leq \|f\|_{\mathcal{H}^2} \left( \sum_{n \geq 1} \frac{1}{n^{2\sigma}} \right)^{\frac{1}{2}} = \|f\|_{\mathcal{H}^2} \sqrt{\zeta(2\sigma)}. \quad \blacksquare$$

Corollary 3.2.3. For $s_0 \in \mathbb{C}_{\frac{1}{2}}$, the induced point evaluation linear functional

$$L_{s_0} : \mathcal{H}^2 \to \mathbb{C}, \quad f \mapsto f(s_0)$$

is bounded and the reproducing kernel at $s_0$ has the form

$$K_{s_0}(s) = \zeta(s + s_0).$$

Proof. Suppose $f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \in \mathcal{H}^2$

$$L_{s_0}(f) = \sum_{n \geq 1} \frac{a_n}{n^{s_0}} = \sum_{n \geq 1} \frac{a_n}{n^{s_0}} = \langle f(s), \zeta(s + s_0) \rangle.$$ 

This completes the proof. \quad \blacksquare

Theorem 3.2.3. $\mathcal{H}^\infty \subset \mathcal{H}^2$, moreover for every $f \in \mathcal{H}^\infty$

$$\|f\|_{\mathcal{H}^2} \leq \|f\|_{\mathcal{H}^\infty}.$$
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**Proof.** For $f(s) = \sum \frac{a_n}{n^s} \in \mathcal{H}^\infty$, by the Corollary 3.1.3. and the Corollary 3.2.2. for every $\epsilon > 0$

$$\sum_{n \geq 1} \frac{|a_n|^2}{n^{2\epsilon}} = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(\epsilon + it)|^2 \, dt \leq \|f\|^2_{\mathcal{H}^\infty}. $$

the proof follows from the monotone convergence theorem for the counting measure. ■

**Corollary 3.2.4.** Let $f$ be a Dirichlet polynomial, then

$$\|f\|^2 = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^2 \, dt.$$

3.3 Hardy Spaces of $\mathbb{T}^\infty$ and the Bohr-Lift

**Definition 3.3.1.** We denote by $\mathbb{T}^\infty$ the product of countable many copies of the unit circle $\mathbb{T}$

$$\mathbb{T}^\infty = \{(z_1, z_2, z_3, \ldots) : \{z_i\}_{i \geq 1} \subset \mathbb{T}\}. $$

$\mathbb{T}^\infty$ is an abelian group with multiplication in each coordinate and by Tychonoff’s theorem it is compact with respect to the product topology. There exists a unique rotation invariant, probability, regular, Borel measure $\mu$, the Haar measure of $\mathbb{T}^\infty$ [27]. We identify the measure $\mu$ with the ordinary probability product measure on $\mathbb{T} \times \mathbb{T} \times \ldots$, i.e. if $B_1, B_2, \ldots, B_n$ are Borel subsets of $\mathbb{T}$ and

$$B = \mathbb{T} \times \mathbb{T} \times \ldots \times B_1 \times \mathbb{T} \times \ldots \times B_2 \times \ldots \times B_n \times \mathbb{T} \times \mathbb{T} \times \mathbb{T} \ldots, $$

then $\mu(B) = \prod_{i=1}^{n} m(B_i)$, where $m$ is the standard normalized Lebesgue measure of $\mathbb{T}$. We will denote by $L^p(\mathbb{T}^\infty)$, $p > 0$ the $L^p$ space with respect to the Haar measure of $\mathbb{T}^\infty$.

**Definition 3.3.2.** We define the sets

$$\mathbb{N}_0^\infty = \{a = (a_1, a_2, \ldots) : \{a_n\}_{n \geq 1} \subset \mathbb{N} \cup \{0\} \text{ and } a \in c_0\}, $$

$$\mathbb{Z}_0^\infty = \{a = (a_1, a_2, \ldots) : \{a_n\}_{n \geq 1} \subset \mathbb{Z} \text{ and } a \in c_0\}. $$

We introduce the multi-index notation, if $z = (z_1, z_2, \ldots) \in \mathbb{T}^\infty$ and $a = (a_1, a_2, \ldots) \in \mathbb{Z}_0^\infty$, then

$$z^a := z_1^{a_1} z_2^{a_2} z_3^{a_3} \ldots \text{ and } z^{-a} = z_1^{-a_1} z_2^{-a_2} z_3^{-a_3} \ldots.$$
Let \( p_1, p_2, p_3, \ldots \) be the increasing sequence of primes. By the fundamental theorem of arithmetic each natural number \( n \in \mathbb{N} \) has a unique factorization
\[
n = \prod_{i \geq 1} p_i^{\gamma_i},
\]
where \( \gamma(n) = (\gamma_1, \gamma_2, \ldots) \in \mathbb{N}_0^\infty \) is the sequence of exponents appearing in the prime number factorization of \( n \). The map \( \gamma : \mathbb{N} \to \mathbb{N}_0^\infty, n \mapsto \gamma(n) \), is a bijection. If we consider \( \mathbb{N}, \mathbb{N}_0^\infty \) as monoids equipped respectively with the standard multiplication and the addition in each coordinate, then \( \gamma \) is an isomorphism.

**Proposition 3.3.1.** The set of trigonometric polynomials on \( \mathbb{T}^\infty \),
\[
M = \text{span} \{ z^a : a \in \mathbb{Z}_0^\infty \},
\]
is dense in the space of continuous complex valued functions \( C(\mathbb{T}^\infty) \) on \( \mathbb{T}^\infty \) with respect to the supremum norm.

**Proof.** We observe that \( M \) is a subalgebra of \( C(\mathbb{T}^\infty) \), closed under conjugation that separates points. The proof follows from the Stone-Weierstrass theorem. \( \blacksquare \)

**Proposition 3.3.2.** The set \( M \) of trigonometric polynomials on \( \mathbb{T}^\infty \) is dense in \( L^p(\mathbb{T}^\infty), \ p \geq 1 \).

**Proof.** By the Proposition 3.3.1. it is sufficient to prove that \( C(\mathbb{T}^\infty) \) is dense in \( L^p(\mathbb{T}^\infty) \). By the regularity of the Haar measure \( \mu \) and the Urysohn’s lemma, follows that every step-function can be approximated by a sequence of continuous functions with respect to the \( L^p \)-norm. It is well known that step-functions are dense in \( L^p(\mathbb{T}^\infty) \), this completes the proof. \( \blacksquare \)

**Definition 3.3.3.** For \( f \in L^1(\mathbb{T}^\infty) \) and \( a \in \mathbb{Z}_0^\infty \), the Fourier coefficient of the integrable function \( f \) at \( a = (a_1, a_2, \ldots) \) is defined by
\[
\hat{f}(a) = \int_{\mathbb{T}^\infty} f(z) z^a \, d\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \cdots \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta_1}, e^{i\theta_2}, \ldots) e^{-a_1i\theta_1} e^{-a_2i\theta_2} \ldots d\theta_1d\theta_2 \ldots .
\]
We denote by \( \hat{f}(n) \) the Fourier coefficient of \( f \) at the sequence \( \gamma(n) \) of exponents appearing in the prime number factorization of \( n \in \mathbb{N} \)
\[
\hat{f}(n) = \int_{\mathbb{T}^\infty} f(z) z^{\gamma(n)} \, d\mu(z) = \hat{f}(\gamma(n)).
\]
For \( N \in \mathbb{N} \), the Poisson kernel in \( N \) variables is defined by
\[
P_N(w, rz) = \prod_{i=1}^{N} \frac{1 - |r_i|^2}{|w_i - r_i z_i|^2}, \text{ where } z, w \in \mathbb{T}^\infty, \ r \in [0, 1)^\infty = [0, 1) \times [0, 1) \times \ldots .
\]
For a function \( f \in L^1(\mathbb{T}^\infty) \), the \( N \)th Poisson integral of \( f \) is defined by
\[
P_N[f](rz) = \int_{\mathbb{T}^\infty} P_N(w, rz) f(w) \, d\mu(w) = \sum_{a \in \mathbb{Z}_0^N} \hat{f}(a)r^{\langle a \rangle} z^a, \ |a| = (|a_1|, |a_2|, \ldots) .
\]
Lemma 3.3.1. Let \( f \in L^p(\mathbb{T}^\infty) \), \( p \in [1, +\infty) \) and \( r \in [0, 1)^\infty \). Then,
\[
\|P_N[f](rz)\|_{L^p(\mathbb{T}^\infty)} \leq \|f(z)\|_{L^p(\mathbb{T}^\infty)}.
\]

Proof. For \( p = 1 \) and \( p = \infty \) the proof is trivial.
For \( p > 1 \), by Hölder’s inequality
\[
|P_N[f](rz)| = \left| \int_{\mathbb{T}^\infty} P_N(w, rz)f(w)d\mu(w) \right|
\leq \left( \int_{\mathbb{T}^\infty} P_N(w, rz)d\mu(w) \right)^{\frac{1}{p}} \left( \int_{\mathbb{T}^\infty} |f(w)|^p d\mu(w) \right)^{\frac{1}{p}}
= \left( \int_{\mathbb{T}^\infty} P_N(w, rz)|f(w)|^p d\mu(w) \right)^{\frac{1}{p}}.
\]
By Fubini’s theorem
\[
\|P_N[f](rz)\|_{L^p(\mathbb{T}^\infty)}^p \leq \int_{\mathbb{T}^\infty} |f(w)|^p \int_{\mathbb{T}^\infty} P_N(w, rz)d\mu(z)d\mu(w) = \|f(z)\|_{L^p(\mathbb{T}^\infty)}^p.
\]

Lemma 3.3.2. Let \( f \in L^p(\mathbb{T}^N) \), \( p \in [1, +\infty) \). Then,
\[
\lim_{r_i \to 1^- , i = 1, \ldots, N} P_N[f](rz) = f(z) \text{ in } L^p(\mathbb{T}^N).
\]

Proof. For every \( \epsilon > 0 \) there exists a trigonometric polynomial (in \( N \) variables) \( P \in M \) such that
\[
\|f - P\|_{L^p(\mathbb{T}^\infty)} < \epsilon.
\]
We observe that
\[
\lim_{r_i \to 1^- , i = 1, \ldots, N} P_N[P](rz) = P(z) \text{ in } L^p(\mathbb{T}^N).
\]
By the Lemma 3.3.1,
\[
\|f(z) - P_N[f](rz)\|_{L^p(\mathbb{T}^\infty)} \leq \|f(z) - P(z)\|_{L^p(\mathbb{T}^\infty)} + \|P(z) - P_N[P](rz)\|_{L^p(\mathbb{T}^\infty)}
+ \|P_N[f - P](rz)\|_{L^p(\mathbb{T}^\infty)}
\leq 2\epsilon + \|P(z) - P_N[P](rz)\|_{L^p(\mathbb{T}^\infty)}.
\]
Thus,
\[
\lim_{r_i \to 1^- , i = 1, \ldots, N} P_N[f](rz) = f(z) \text{ in } L^p(\mathbb{T}^N).
\]
\[\blacksquare\]
**Definition 3.3.4.** For $f \in L^1(\mathbb{T}^\infty)$ and $N \in \mathbb{N}$. We define the function $f_N : \mathbb{T}^N \subset \mathbb{T}^\infty \to \mathbb{C}$ as

$$f_N(w) = f_N(w_1, \ldots, w_N) = \int_{\mathbb{T}^\infty} f(w, z) d\mu(z).$$

**Lemma 3.3.3.** Suppose $f \in L^p(\mathbb{T}^\infty)$, $p \in [1, +\infty)$ and $f_N, N \in \mathbb{N}$ is as above.

1. $f_N \in L^p(\mathbb{T}^\infty)$ and $\|f_N\|_{L^p(\mathbb{T}^\infty)} \leq \|f\|_{L^p(\mathbb{T}^\infty)}$.
2. $\hat{f}_N(a) = \hat{f}(a), \ \forall a \in \mathbb{Z}^N$ and $\hat{f}_N(a) = 0, \ \forall a \notin \mathbb{Z}^N$.
3. $\lim_{N \to +\infty} f_N = f$ in $L^p(\mathbb{T}^\infty)$.

**Proof.**

1. For $p = 1$ the proof is trivial.

For $p > 1$, by Hölder’s inequality

$$\|f_N\|_{L^p(\mathbb{T}^\infty)}^p = \int_{\mathbb{T}^\infty} \left( \int_{\mathbb{T}^\infty} f(w, z) d\mu(z) \right)^p d\mu(w)$$

$$\leq \int_{\mathbb{T}^\infty} \left( \int_{\mathbb{T}^\infty} d\mu(z) \right)^{\frac{p}{q}} \int_{\mathbb{T}^\infty} |f(w, z)|^p d\mu(z) d\mu(w)$$

$$= \|f\|_{L^p(\mathbb{T}^\infty)}^p.$$

2. By Fubini’s theorem

$$\hat{f}_N(a) = \int_{\mathbb{T}^\infty} \int_{\mathbb{T}^\infty} f(w, z) \overline{w}^a d\mu(w) d\mu(z) = \hat{f}(a) \text{ if } a \in \mathbb{Z}^N \text{ and } \hat{f}_N(a) = 0, \ \text{if } a \notin \mathbb{Z}^N.$$

3. For every $\epsilon > 0$ there exists a trigonometric polynomial $P \in M$ such that

$$\|f - P\|_{L^p(\mathbb{T}^\infty)} < \epsilon.$$

We observe that $P_N = P$ for $N$ sufficiently large, so

$$\|f - f_N\|_{L^p(\mathbb{T}^\infty)} \leq \|f - P\|_{L^p(\mathbb{T}^\infty)} + \|P - P_N\|_{L^p(\mathbb{T}^\infty)} + \|(f - P)_N\|_{L^p(\mathbb{T}^\infty)} \leq 2\epsilon.$$

The proof follows.
Theorem 3.3.1. Let \( f \in L^p(\mathbb{T}^\infty), \ p \in [1, +\infty), \) then
\[
f(z) \in \left\{ P_N[f](rz) = \sum_{a \in \mathbb{Z}^N} \hat{f}(a) r^{|a|} z^a : \ r \in [0, 1)^\infty, \ N \in \mathbb{N} \right\}.
\]

Proof. By the Lemma 3.3.3. (2. and 3.) and the Lemma 3.3.2. for every \( \epsilon > 0 \) there exists a \( N \in \mathbb{N} \) and a \( r \in [0, 1)^\infty \) such that
\[
\| f - f_N \|_{L^p(\mathbb{T}^\infty)} \leq \frac{\epsilon}{2} \quad \text{and} \quad \| P_N[f_N](rz) - f_N \|_{L^p(\mathbb{T}^\infty)} \leq \frac{\epsilon}{2}.
\]
Thus,
\[
\| P_N[f](rz) - f \|_{L^p(\mathbb{T}^\infty)} \leq \epsilon.
\]
\[
\blacksquare
\]

Corollary 3.3.1. The Fourier transform \( F \) between \( L^1(\mathbb{T}^\infty) \) and the space of formal power series in infinite many variables is injective
\[
F(f) = \sum_{a \in \mathbb{Z}_0^\infty} \hat{f}(a) z^a.
\]

Definition 3.3.5. For \( p \in [1, \infty) \), the Hardy space \( H^p(\mathbb{T}^\infty) \) is defined as the subspace of \( L^p(\mathbb{T}^\infty) \) with "non-negative" Fourier coefficients
\[
H^p(\mathbb{T}^\infty) = \{ f \in L^p(\mathbb{T}^\infty) : \hat{f}(a) = 0, \ \forall a \not\in \mathbb{N}_0^\infty \}.
\]

Corollary 3.3.2. The Hardy space \( H^p(\mathbb{T}^\infty), \ p \in [1, +\infty) \) is equal to the closure of analytic polynomials in \( L^p(\mathbb{T}^\infty) \)
\[
H^p(\mathbb{T}^\infty) = \text{span} \{ z^a : a \in \mathbb{N}_0^\infty \} = \text{span} \{ z^{\gamma(n)} : n \in \mathbb{N} \}.
\]

Moreover, if \( f \in H^p(\mathbb{T}^\infty), \) then
\[
f(z) \in \left\{ P_N[f](rz) = \sum_{a \in \mathbb{Z}_0^N} \hat{f}(a) r^{|a|} z^a : \ r \in [0, 1)^\infty, \ N \in \mathbb{N} \right\}.
\]

The Fourier series of a function \( f \in H^p(\mathbb{T}^\infty) \) can be written in the forms
\[
\sum_{a \in \mathbb{N}_0^\infty} \hat{f}(a) z^a \quad \text{or} \quad \sum_{n \geq 1} \hat{f}(n) z^{\gamma(n)}.
\]

The Hardy space \( H^2(\mathbb{T}^\infty) \) is a separable Hilbert space equipped with the inner product
\[
\langle f, g \rangle = \sum_{n \geq 1} \hat{f}(n) \overline{\hat{g}(n)} = \sum_{a \in \mathbb{N}_0^\infty} \hat{f}(a) \overline{\hat{g}(a)}.
\]

The standard orthonormal basis has the forms:
\[
\{ z^{\gamma(n)} : n \in \mathbb{N} \} = \{ z^a : a \in \mathbb{N}_0^\infty \}.
\]
Definition 3.3.6. Let \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \) be a Dirichlet series. We substitute every \( n \in \mathbb{N} \) with its prime factorization and we consider every factor \( \frac{1}{p_k^s} \) as a new variable. Then, the Dirichlet series \( f \) corresponds to a formal power series \( \mathcal{B}(f) \) in infinitely many variables, called the **Bohr-lift** of \( f \)

\[
f(s) = \sum_{n \geq 1} a_n \left( \frac{1}{p_1^s} \right)^{\gamma_1} \left( \frac{1}{p_2^s} \right)^{\gamma_2} \cdots \\
2^{-s} \mapsto z_1, \ 3^{-s} \mapsto z_2, \ 5^{-s} \mapsto z_3, \ldots \\
\mathcal{B}(f)(z) = \sum_{n \geq 1} a_n z_1^{\gamma_1} z_2^{\gamma_2} \cdots = \sum_{n \geq 1} a_n z^{\gamma(n)}.
\]

The Bohr-lift defines an isomorphism between the space of formal Dirichlet series and the space of formal power series in infinitely many variables.

Theorem 3.3.2. The Bohr-lift is an isometric isomorphism between the Hilbert spaces \( \mathcal{H}^2 \) and \( H^2(\mathbb{T}^\infty) \).

**Proof.** Let \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \in \mathcal{H}^2 \), then \( \mathcal{B}(f)(z) = \sum_{n \geq 1} a_n z^{\gamma(n)} \). We consider the partial sums

\[
S_N(z) = \sum_{n=1}^{N} a_n z^{\gamma(n)}.
\]

By the Parseval’s formula \( \{S_N\}_{N \geq 1} \) is a Cauchy’s sequence in \( H^2(\mathbb{T}^\infty) \). This implies that \( \mathcal{B}(f) \in H^2(\mathbb{T}^\infty) \) and \( \|\mathcal{B}(f)\|_{H^2(\mathbb{T}^\infty)} = \|f\|_{\mathcal{H}^2} \). \( \mathcal{B} \) is injective, so it is sufficient to prove that \( \mathcal{B} : \mathcal{H}^2 \to H^2(\mathbb{T}^\infty) \) is surjective. For \( g \in H^2(\mathbb{T}^\infty) \), we observe that \( f(s) = \sum_{n \geq 1} \frac{\gamma(n)}{n^s} \in \mathcal{H}^2 \) and \( \mathcal{B}(f) = g \).

Lemma 3.3.4. **(Kronecker)** Let \( x_1, x_2, x_3, \ldots, x_n \in \mathbb{R} \) be linear independent over \( \mathbb{Q} \). Then, for every vector \( \vec{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n \) and for every \( \epsilon > 0 \) there exists a positive number \( t > 0 \) and a vector \( \vec{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n \) such that

\[
|tx_k - y_k - b_k| < \epsilon, \ k = 1, \ldots, n.
\]

**Proof.** We consider the function

\[
f(t) = 1 + \sum_{k=1}^{n} e^{2\pi i (tx_k - y_k)}.
\]

It is sufficient to prove that \( \sup \{ |f(t)| : t \in \mathbb{R} \} = n + 1 \). We observe that \( |f(t)| \leq n + 1, \ \forall t \in \mathbb{R} \) and we assume that \( \sup \{|f(t)| : t \in \mathbb{R} \} = L \in (0, n+1) \). Suppose \( m \in \mathbb{N} \) and

\[
f^m(t) = \sum_{r_0 + r_1 + \ldots + r_n = m} a_r e^{2\pi i (r,x)}, \ x = (1, x_1, \ldots, x_n), \ r = (r_0, r_1, \ldots, r_n) \in (\mathbb{Z}_{\geq 0})^{n+1}.
\]
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It is easy to prove that

$$a_r = \frac{m!}{r_0!r_1!\ldots r_n!} e^{-2\pi i (r,y)}, \quad y = (1, y_1, y_2, \ldots, y_n).$$

By the independence of $x_1$, ..., $x_n$ follows that if $r_1 \neq r_2$ then, $e^{2\pi it(r_1, x)} \neq e^{2\pi it(r_2, x)}$. Thus,

$$|a_r| = \left| \lim_{T \to +\infty} \frac{1}{T} \int_0^T f^m(t)e^{-2\pi it(r,x)} dt \right| \leq L^m. \quad (3.14)$$

We observe that

$$\sum_{r_0+\ldots+r_n=m} |a_r| = (n+1)^m$$

and by (3.14)

$$(n+1)^m \leq \binom{m+n}{n} L^m$$

$$\lim_{m \to +\infty} \frac{n!m!}{(m+n)!} \leq \lim_{m \to +\infty} \left( \frac{L}{n+1} \right)^m = 0, \text{ contradiction.}$$

\[ \blacksquare \]

**Corollary 3.3.3.** Let $d \in \mathbb{N}$, then the set \{(p_1^{-it}, p_2^{-it}, \ldots, p_d^{-it}) : t \in \mathbb{R}\} is dense in $\mathbb{T}^d$.

**Proof.** We will prove that $-\log(p_1)$, ..., $-\log(p_d)$ are linear independent over $\mathbb{Q}$. For $\lambda_1$, ..., $\lambda_d \in \mathbb{Q}$ such that

$$-\lambda_1 \log(p_1) - \ldots - \lambda_N \log(p_d) = 0$$

$$p_1^{\lambda_1} \ldots p_d^{\lambda_d} = 1,$$

by the fundamental theorem of arithmetic

$$\lambda_1 = \ldots = \lambda_d = 0.$$

By the Lemma 3.3.4. for every $(e^{iy_1}, \ldots, e^{iy_d}) \in \mathbb{T}^d$ and $\varepsilon > 0$ there exists a $t \in \mathbb{R}$ such that

$$|p_k^{-it} - e^{iy_k}| < \varepsilon, \quad k = 1, \ldots, d.$$

\[ \blacksquare \]

**Lemma 3.3.5.** Let $P(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$ be a Dirichlet polynomial. Then,

$$\|P\|_{H^\infty} = \|\mathcal{R}(P)\|_{H^\infty(\mathbb{T}^\infty)}.\]
**Proof.** There exists a $d \in \mathbb{N}$ such that $p_d \leq N < p_{d+1}$. The Bohr-lift of $P$ is continuous on $\mathbb{T}^d$

$$\mathcal{B}(P)(z) = \sum_{n=1}^{N} a_n z_{1}^{\gamma_1(n)} \cdots z_{d}^{\gamma_d(n)}.$$

By the Corollary 3.3.3.

$$\|\mathcal{B}(P)\|_{H^\infty(\mathbb{T}^\infty)} = \max_{\mathbb{T}^d} |\mathcal{B}(P)(z_1, \ldots, z_d)|$$

$$= \sup_{\mathbb{R}} \left| \mathcal{B}(P)(p_1^{-it}, \ldots, p_d^{-it}) \right|$$

$$= \sup_{\mathcal{R}} |P(it)| = \|P\|_{\mathcal{H}^\infty}.$$

\[\blacksquare\]

**Theorem 3.3.3.** The Bohr-lift is an isometric isomorphism between $\mathcal{H}^\infty$ and $H^\infty(\mathbb{T}^\infty)$.

**Proof.** Suppose $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \in \mathcal{H}^\infty$ and $S_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$. Let $\delta > 0$, then for every $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$\|f(s + \delta) - S_N(s + \delta)\|_{\mathcal{H}^\infty} < \epsilon, \; N \geq n_0.$$

By the Lemma 3.3.5.

$$\left| \sum_{n=1}^{N} \frac{a_n}{n^\delta} z^{\gamma(n)} \right| \leq \|f(s + \delta)\|_{\mathcal{H}^\infty} + \epsilon \leq \|f\|_{\mathcal{H}^\infty} + \epsilon, \; z \in \mathbb{T}^\infty, \; N \geq n_0.$$

The series $\sum_{n=1}^{\infty} a_n z^{\gamma(n)}$ converges almost everywhere since $\mathcal{B}(f) \in H^2(\mathbb{T}^\infty)$, letting $N \to +\infty$, $\epsilon \to 0$ and then by Fatou’s lemma letting $\delta \to 0$ we obtain that

$$\left| \sum_{n=1}^{\infty} a_n z^{\gamma(n)} \right| \leq \|f\|_{\mathcal{H}^\infty} \quad \text{almost everywhere.}$$

This implies that

$$\|\mathcal{B}(f)\|_{H^\infty(\mathbb{T}^\infty)} \leq \|f\|_{\mathcal{H}^\infty}. \quad (3.15)$$

For $g \in H^\infty(\mathbb{T}^\infty)$ and $N \in \mathbb{N}$, we observe that

$$\|P_N[g](rz)\|_{H^\infty(\mathbb{T}^\infty)} \leq \|g\|_{H^\infty(\mathbb{T}^\infty)}.$$

We consider the Dirichlet series $f(s) = \mathcal{B}(g)^{-1} = \sum_{n=1}^{\infty} \frac{\hat{g}(n)}{n^s}$. 

3.3. HARDY SPACES OF $\mathbb{T}^\infty$ AND THE BOHR-LIFT

\begin{align*}
|f(s)| & \leq \limsup_{N \to +\infty} \limsup_{r_i \to 1^-} \left\| \sum_{n=1}^{N} \frac{g(n)r^{\gamma(n)}}{n^{s}} \right\|_{\mathcal{H}^{\infty}} \\
& \leq \limsup_{N \to +\infty} \limsup_{r_i \to 1^-} \| P_N(g)(rz) \|_{H^{\infty}(\mathbb{T}^\infty)} \\
& \leq \| g \|_{H^{\infty}(\mathbb{T}^\infty)}.
\end{align*}

Thus,

$$
\| f \|_{\mathcal{H}^{\infty}} \leq \| g \|_{H^{\infty}(\mathbb{T}^\infty)}. \tag{3.16}
$$

By (3.15) and (3.16) follows that the Bohr-lift is an isometric isomorphism between $\mathcal{H}^{\infty}$ and $H^{\infty}(\mathbb{T}^\infty)$.

\textbf{Theorem 3.3.4.} Let $f(s) = \sum_{n \geq 1} \frac{a_n}{n}$ be a Dirichlet polynomial and $p \in [1, +\infty)$. Then,

$$
\| f \|_{\mathcal{H}^{p}} := \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^p dt = \| B(f) \|_{H^{p}(\mathbb{T}^\infty)}^{p}.
$$

\textbf{Proof.} We observe that if $g_1, g_2$ are two formal Dirichlet series, then

$$
B(g_1g_2) = B(g_1)B(g_2).
$$

By Parseval's formula for Dirichlet series

$$
\| f^n \|_{\mathcal{H}^{2}} := \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^{2n} dt = \| B(f^n) \|_{H^{2}(\mathbb{T}^\infty)}^{2} = \int_{\mathbb{T}^\infty} |B(f)|^{2n} d\mu, \ \forall n \in \mathbb{N}.
$$

Suppose $C := \sum_{n \geq 1} |a_n| < +\infty$, by the Weierstrass approximation theorem for every $\epsilon > 0$ there exists a real polynomial $P(x) = \sum_{k=0}^{n} b_k x^k$ such that

$$
| x^p - P(x) | < \epsilon, \ \forall x \in [0, C].
$$
For $T > 0$ sufficiently large
\[
\left| \frac{1}{2T} \int_{-T}^{T} |f(it)|^p dt - \int_{\mathbb{T}^\infty} |\mathcal{B}(f)|^p d\mu \right| \leq \left| \frac{1}{2T} \int_{-T}^{T} P\left(|f(it)|^2\right) dt - \int_{\mathbb{T}^\infty} |\mathcal{B}(f)|^p d\mu \right| + \epsilon
\]
\[
= \sum_{k=0}^{n} b_k \left[ \frac{1}{2T} \int_{-T}^{T} |f(it)|^{2k} dt - \int_{\mathbb{T}^\infty} |\mathcal{B}(f)|^p d\mu \right] + \epsilon
\]
\[
\leq \sum_{k=0}^{n} b_k \left[ \int_{\mathbb{T}^\infty} |\mathcal{B}(f)|^{2k} d\mu - \int_{\mathbb{T}^\infty} |\mathcal{B}(f)|^p d\mu \right] + 2\epsilon
\]
\[
= \left| \int_{\mathbb{T}^\infty} P\left(|\mathcal{B}(f)|^2\right) d\mu - \int_{\mathbb{T}^\infty} |\mathcal{B}(f)|^p d\mu \right| + 2\epsilon
\]
\[
\leq 3\epsilon.
\]

Letting $T \to +\infty$ and then $\epsilon \to 0$, we obtain that
\[
\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^p dt = \|\mathcal{B}(f)\|_{H^p(\mathbb{T}^\infty)}^p.
\]

\[\|\cdot\|_{\mathcal{H}^p}\] is a norm on the space of Dirichlet polynomials.

**Definition 3.3.7.** For $p \in [1, +\infty)$, the Hardy space $\mathcal{H}^p$ of Dirichlet series is defined as the completion of Dirichlet polynomials with respect to the $\|\cdot\|_{\mathcal{H}^p}$ norm.

By the Corollary 3.3.2. the Hardy space $H^p(\mathbb{T}^\infty)$ is the completion of analytic polynomials in $L^p(\mathbb{T}^\infty)$ and by the Theorem 3.3.4. the Bohr-lift is an isometric isomorphism between the spaces of Dirichlet polynomials and analytic polynomials. This implies that the Bohr-lift is an isometric isomorphism between $\mathcal{H}^p$ and $H^p(\mathbb{T}^\infty)$

\[\mathcal{H}^p \simeq H^p(\mathbb{T}^\infty), \ p \geq 1.\]
Chapter 4

Multiplicative Hankel Operators

H. Helson [13] introduced the Multiplicative Hankel Operators or Helson matrices, asked if the analogue of the Nehari theorem holds [14] and proved that every Multiplicative Hankel Operator \( H_\psi \) of Hilbert-Schmidt type admits bounded symbol \( \phi \in L^\infty(\mathbb{T}^\infty) \). J. Ortega-Cerdà and K. Seip [20] gave a negative answer for the general question, they proved the existence of bounded multiplicative Hankel operators without bounded symbols and O. F. Brevig found an explicit counterexample [2]. O. F. Brevig and K. M. Perfekt [3] proved that for \( p > p_0 = (1 - \frac{\log \pi}{\log 4})^{-1} \) there exist multiplicative Hankel operators in the Schatten class \( S_p \) which do not arise from bounded symbols and we will construct an explicit counterexample of a multiplicative Hankel operator in \( S_p, \ p > p_0 \) without bounded symbols.

**Definition 4.0.1.** An infinite matrix with complex entries that depend only on the product of coordinates called a multiplicative Hankel matrix. Thus, an arbitrary multiplicative Hankel matrix has the following form:

\[
M_a = \begin{pmatrix}
  a_1 & a_2 & a_3 & a_4 & \cdots \\
  a_2 & a_4 & a_6 & a_8 & \cdots \\
  a_3 & a_6 & a_9 & a_{12} & \cdots \\
  a_4 & a_8 & a_{12} & a_{16} & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}
\]

We observe that a sequence of complex numbers \( a = \{a_n\}_{n \geq 1} \) corresponds to a multiplicative Hankel matrix \( M_a = [a_{ij}]_{i,j \geq 1} \).

For a function \( g \in L^1(\mathbb{T}^\infty) \), the sequence of "non-negative" Fourier coefficients \( \hat{g}(n) \) for \( n \geq 1 \) induces a multiplicative Hankel matrix.
\[ M_g = \begin{pmatrix} \hat{g}(1) & \hat{g}(2) & \hat{g}(3) & \hat{g}(4) & \cdots \\ \hat{g}(2) & \hat{g}(4) & \hat{g}(6) & \hat{g}(8) & \cdots \\ \hat{g}(3) & \hat{g}(6) & \hat{g}(9) & \hat{g}(12) & \cdots \\ \hat{g}(4) & \hat{g}(8) & \hat{g}(12) & \hat{g}(16) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

Similarly, a Dirichlet series \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \) induces the multiplicative Hankel matrix \( M_a = [a_{ij}]_{i,j \geq 1} \).

**Definition 4.0.2.** Suppose \( g \in L^2(\mathbb{T}^\infty) \) and \( M_g \) is the induced multiplicative Hankel matrix. \( M_g \) induces an operator on the dense subset of analytic polynomial in \( H^2(\mathbb{T}^\infty) \), that we will denote by \( H_g \). Let \( P(z) = \sum n \hat{P}(n)z^{\gamma(n)} \) be an analytic polynomial, then

\[
H_g(P) = \sum_{n \in \mathbb{N}} c_n z^{\gamma(n)}, \quad \text{where} \quad c_n = \sum_{m \in \mathbb{N}} \hat{g}(\gamma(n) + \gamma(m))\hat{P}(m) = \sum_{m \in \mathbb{N}} \hat{g}(nm)\hat{P}(m),
\]

or equivalently

\[
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} \hat{g}(1) & \hat{g}(2) & \hat{g}(3) & \hat{g}(4) & \cdots \\ \hat{g}(2) & \hat{g}(4) & \hat{g}(6) & \hat{g}(8) & \cdots \\ \hat{g}(3) & \hat{g}(6) & \hat{g}(9) & \hat{g}(12) & \cdots \\ \hat{g}(4) & \hat{g}(8) & \hat{g}(12) & \hat{g}(16) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \hat{P}(1) \\ \hat{P}(2) \\ \hat{P}(3) \\ \hat{P}(4) \\ \vdots \end{bmatrix}
\]

Using the Minkowski inequality it is easy to prove \( H_g \) is well defined, i.e. \( H_g(P) \in H^2(\mathbb{T}^\infty) \).

We consider the operator \( H_g \) acting on the basis \( \{z^a : a \in \mathbb{N}_0^\infty\} \),

\[
H_g(P) = \sum_{a \in \mathbb{N}_0^\infty} c_a z^a, \quad \text{where} \quad c_a = \sum_{b \in \mathbb{N}_0^\infty} \hat{g}(a+b)\hat{P}(b).
\]

The operator \( H_g \) is a **multiplicative Hankel operator** if it has an extension on \( H^2(\mathbb{T}^\infty) \). In this case, the function \( g \) called a **symbol** of \( H_g \).

**Definition 4.0.3.** Suppose \( f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \in \mathcal{H}^2 \), \( f \) corresponds to a multiplicative Hankel matrix \( M_a \) and induces an operator \( H_f \) defined on the dense subset of Dirichlet polynomial with range in \( \mathcal{H}^2 \). Let \( P(s) = \sum_{n \geq 1} \frac{b_n}{n^s} \) be a Dirichlet polynomial, then

\[
H_f(P) = \sum_{n \geq 1} c_n n^s, \quad \text{where} \quad c_n = \sum_{m \in \mathbb{N}} a_{nm} b_m,
\]

or equivalently
The operator $H_f$ is a **multiplicative Hankel operator** if it has an extension on $H^2$. Under the Bohr-lift every multiplicative Hankel operator on $H^2(T^\infty)$ corresponds to a multiplicative Hankel operator on $H^2$. We can investigate the properties of multiplicative Hankel operators, in each of those spaces, and obtain common results.

**Proposition 4.0.1.** Suppose $\psi \in L^2(T^\infty)$ and $H_\psi$ is the induced operator. If $f$, $g$ are analytic polynomials, then

$$\langle H_\psi(f), g \rangle = \langle \psi(z), \overline{f(z)} g(z) \rangle,$$

where $\gamma = \sum_{n \geq 1} \hat{f}(n)z^{-n}$.

**Proof.** We observe that $\gamma(d) = \gamma(n) - \gamma(d)$, $d | n$, $n \in \mathbb{N}$.

$$\langle H_\psi(f), g \rangle = \sum_{n \geq 1} \sum_{m \geq 1} \hat{\psi}(nm) \overline{\hat{f}(m)} \overline{\hat{g}(n)} = \sum_{n \geq 1} \hat{\psi}(n) \sum_{d | n} \hat{f}(d) \overline{\hat{g}(\frac{n}{d})} = \langle \psi(z), \overline{f(z)} g(z) \rangle.$$

**Corollary 4.0.1.** Suppose $\psi \in H^2$ and $H_\psi$ is the induced operator. If $f$, $g$ are Dirichlet polynomials, then

$$\langle H_\psi(f), g \rangle = \langle \psi(s), \overline{f(s)} g(s) \rangle.$$

As we saw in the proof of the Nehari theorem, the above inner product equality holds for additive Hankel operators on $H^2(D)$ and some authors used this to define the analogue of the additive Hankel operators on $H^2$.

**Theorem 4.0.1.** Let $\psi \in L^\infty(T^\infty)$ and $H_\psi$ be the induced operator. Then, $H_\psi$ is a bounded multiplicative Hankel operator on $H^2(T^\infty)$ and $\|H_\psi\| \leq \|\psi\|_{L^\infty(T^\infty)}$.

**Proof.** For an analytic polynomial $P$ on $T^\infty$,

$$\|H_\psi(P)\|_{H^2(T^\infty)}^2 = \langle H_\psi(P), H_\psi(P) \rangle$$

$$= \langle \psi(z), \overline{P(z)} H_\psi(P) \rangle$$

$$= \int_{T^\infty} \psi(z) P(\overline{z}) H_\psi(P)(z) d\mu(z)$$

$$\leq \|\psi\|_{L^\infty(T^\infty)} \|P\|_{H^2(T^\infty)} \|H_\psi(P)\|_{H^2(T^\infty)}.$$
By the density of analytic polynomials on $H^2(\mathbb{T}^\infty)$ we obtain that $H_\psi$ is a bounded multiplicative Hankel operator with norm $\|H_\psi\| \leq \|\psi\|_{L^\infty(\mathbb{T}^\infty)}$. ■

**Lemma 4.0.1. (Carleman)** Let $f \in H^1(\mathbb{H})$, then

$$\left(\sum_{n \geq 0} \frac{|\hat{f}(n)|^2}{n + 1}\right)^{\frac{1}{2}} \leq \|f\|_{H^1}.$$  

**Proof.** By the canonical factorization theorem there exist $g_1, g_2 \in H^2(\mathbb{H})$ such that $\hat{f} = g_1 g_2$ and $\|f\|_{H^2} = \|g_1\|_{H^2} \|g_2\|_{H^2}$. By the Cauchy-Schwarz inequality

$$|\hat{f}(n)|^2 = \left|\sum_{k=0}^{n} \hat{g}_1(k) \hat{g}_2(n-k)\right|^2 \leq (n+1) \sum_{k=0}^{n} |\hat{g}_1(k)|^2 |\hat{g}_2(n-k)|^2, \quad \forall n \in \mathbb{N},$$

$$\sum_{n \geq 0} \frac{|\hat{f}(n)|^2}{n + 1} \leq \sum_{k \geq 0} |\hat{g}_1(k)|^2 \sum_{n \geq k} |\hat{g}_2(n-k)|^2 \leq \|g_1\|_{H^2}^2 \|g_2\|_{H^2}^2 = \|f\|_{H^1}^2.$$  

Carleman’s inequality implies that the inclusion map between the Hardy space $H^1(\mathbb{H})$ and the Bergman space $A^2$ is a contraction.

**Lemma 4.0.2.** Let $f \in H^1(\mathbb{T}^\infty)$, then

$$\left(\sum_{n \geq 1} \frac{|\hat{f}(n)|^2}{d(n)}\right)^{\frac{1}{2}} \leq \|f\|_{H^1(\mathbb{T}^\infty)},$$

where $d(n)$ is the divisor count function

$$d(n) = d(p_1^{\gamma_1} p_2^{\gamma_2} \cdots) = \prod_{i \geq 1} (\gamma_i + 1), \quad n \in \mathbb{N}.$$  

**Proof.** First we will prove the inequality for functions $f \in H^1(\mathbb{T}^N), \ N \in \mathbb{N}$. Suppose $f(z) \sim \sum_{a \in \mathbb{Z}^N_{\geq 0}} c(a_1, \ldots, a_N) z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}$ is such a function, it is sufficient to prove that

$$\left(\sum_{a \in \mathbb{Z}^N_{\geq 0}} \frac{|c(a_1, \ldots, a_N)|^2}{(a_1 + 1)(a_2 + 1) \cdots (a_N + 1)}\right)^{\frac{1}{2}} \leq \|f\|_{H^1(\mathbb{T}^\infty)}.$$  

We will prove the inequality (4.1) by induction on $N \in \mathbb{N}$. For $N = 1$, it holds by the Lemma 4.0.1. We assume that it holds for $N \in \mathbb{N}$. To prove it for $N + 1$, we define the operators

$$T_n(g)(z) = \sum_{a \in \mathbb{Z}^N_{\geq 0}} \frac{\hat{g}(a)}{\sqrt{a_n + 1}} z_1^{a_1} \cdots z_N^{a_N} z_{N+1}^{a_{N+1}}, \quad n = 1, \ldots, N + 1.$$
We define the linear functional $L$ by the Lemma 4.0.2. and (4.2)

The proof follows from Fatou’s lemma letting

Suppose $f \in H^1(\mathbb{T}^\infty)$ is arbitrary, by the Lemma 3.3.1. and (4.1) for every $N \in \mathbb{N}$ and $r \in [0, 1)^\infty$

The proof follows from Fatou’s lemma letting $r_1, \ldots, r_N \to 1^-$ and then $N \to +\infty$.  

**Theorem 4.0.2. (Helson) Let $H_\psi$, $\psi \in L^2(\mathbb{T}^\infty)$ be a multiplicative Hankel operator on $H^2(\mathbb{T}^\infty)$ of Hilbert-Schmidt type. Then, $H_\psi$ admits bounded symbol $\phi \in L^\infty(\mathbb{T}^\infty)$ and

$$\|\phi\|_{L^\infty(\mathbb{T}^\infty)} \leq \|H_\psi\|_{S_2}.$$**

**Proof.** We observe that

$$\|H_\psi\|_{S_2}^2 = \sum_{n \geq 1} \|H_\psi \left( z^n \right) \|_{H^2(\mathbb{T}^\infty)}^2 = \sum_{n \geq 1} \sum_{m \geq 1} |\psi(nm)|^2 = \sum_{n \geq 1} d(n) |\psi(n)|^2. \quad (4.2)$$

We define the linear functional $L : H^1(\mathbb{T}^\infty) \to \mathbb{C}$ as

$$L(f) = \sum_{n \geq 1} \hat{f}(n) \psi(n).$$

By the Lemma 4.0.2. and (4.2)

$$|L(f)| \leq \sum_{n \geq 1} \frac{|\hat{f}(n)|}{\sqrt{d(n)}} \sqrt{d(n)|\psi(n)|} \leq \left( \sum_{n \geq 1} d(n)|\psi(n)| \right)^{\frac{1}{2}} \left( \sum_{n \geq 1} \frac{|\hat{f}(n)|^2}{d(n)} \right)^{\frac{1}{2}} \leq \|f\|_{H^1(\mathbb{T}^\infty)} \|H_\psi\|_{S_2}.$$
By the Hahn-Banach theorem the linear functional $L$ has an extension $\tilde{L}$ on $L^1(\mathbb{T}^\infty)$ such that $\|\tilde{L}\| \leq \|H_\psi\|_{S_2}$. There exists a bounded function $\phi \in L^\infty(\mathbb{T}^\infty)$ such that

$$\tilde{L}(f) = \int_{\mathbb{T}^\infty} f(z)\phi(\overline{z})d\mu(z) \quad \text{and} \quad \|\phi\|_{L^\infty(\mathbb{T}^\infty)} = \|\tilde{L}\| \leq \|H_\psi\|_{S_2}.$$ 

The Haar measure $\mu$ is rotation invariant, so

$$\psi(n) = L(z^{\gamma(n)}) = \tilde{L}(z^{\gamma(n)}) = \int_{\mathbb{T}^\infty} z^{\gamma(n)}\phi(\overline{z})d\mu(z) = \tilde{\phi}(n), \quad n \geq 1.$$ 

Thus, the function $\phi$ is a bounded symbol for the multiplicative Hankel operator $H_\psi$. \hfill \blacksquare

**Definition 4.0.4.** The weak product $H^2(\mathbb{T}^\infty) \odot H^2(\mathbb{T}^\infty)$ is defined as the Banach space completion of all the finite sums of the form $F(z) = \sum f_i(z)g_i(z)$, where $f_i, g_i \in H^2(\mathbb{T}^\infty)$ with respect to the norm

$$\|F\|_{H^2(\mathbb{T}^\infty) \odot H^2(\mathbb{T}^\infty)} = \inf\{\sum \|f_i(z)\|_{H^2(\mathbb{T}^\infty)} \|g_i(z)\|_{H^2(\mathbb{T}^\infty)} : F(z) = \sum f_i(z)g_i(z)\}.$$ 

**Remark.** Similarly we can define the weak product $H^2(\mathbb{D}) \odot H^2(\mathbb{D})$ and a direct consequence of the canonical factorization theorem is that $H^1(\mathbb{D}) = H^2(\mathbb{D}) \odot H^2(\mathbb{D})$. We observe that the equality $H^1(\mathbb{T}^\infty) = H^2(\mathbb{T}^\infty) \odot H^2(\mathbb{T}^\infty)$ would imply the Nehari theorem for $\mathbb{T}^\infty$, but as we will prove later this is not true. The equality $H^1(\mathbb{T}^d) = H^2(\mathbb{T}^d) \odot H^2(\mathbb{T}^d)$, $d \in (1, +\infty)$ or the Nehari theorem for $\mathbb{T}^d$, $d \in (1, +\infty)$ are open problems.

**Proposition 4.0.2.** The space $BH$ of bounded multiplicative Hankel operators on $H^2(\mathbb{T}^\infty)$ is a Banach space.

**Proof.** For $H_\psi \in BH$, we consider the linear functional

$$L(H_\psi): H^2(\mathbb{T}^\infty) \odot H^2(\mathbb{T}^\infty) \to \mathbb{C},$$

$$L(H_\psi)\left(\sum_{i=1}^{N} f_i g_i\right) = \left(\sum_{i=1}^{N} f_i g_i, \psi\right) = \sum_{i=1}^{N} \langle H_\psi(\overline{f_i(z)}), g_i \rangle, \quad f_i, g_i \in H^2(\mathbb{T}^\infty), \quad N \in \mathbb{N}.$$ 

By the Cauchy-Schwarz inequality

$$\left| L(H_\psi)\left(\sum_{i=1}^{N} f_i g_i\right) \right| \leq \sum_{i=1}^{N} \|H_\psi\| \|f_i\|_{H^2(\mathbb{T}^\infty)} \|g_i\|_{H^2(\mathbb{T}^\infty)},$$

$$\|L(H_\psi)\| \leq \|H_\psi\|.$$
Moreover,
\[ \| H_\psi \| = \sup \{ \langle H_\psi(f), g \rangle : \| f \|_{H^2(T^\infty)} = \| g \|_{H^2(T^\infty)} = 1 \} = \sup \{ \| L(H_\psi)(f) \| : \| f \|_{H^2(T^\infty)} = \| g \|_{H^2(T^\infty)} = 1 \} \leq \sup \{ \| L(H_\psi)(F) \| : \| F \|_{H^2(T^\infty) \odot H^2(T^\infty)} \leq 1 \} = \| L(H_\psi) \|. \] (4.3)

Thus,
\[ \| L(H_\psi) \| = \| H_\psi \|. \] (4.3)

Let \( \Lambda \in (H^2(T^\infty) \odot H^2(T^\infty))^* \), then \( \Lambda|_{H^2(T^\infty)} \in (H^2(T^\infty))^* \) and by the Riesz representation theorem there exists a function \( \psi \in H^2(T^\infty) \) such that
\[ \Lambda(f) = \langle f, \psi \rangle, \ f \in H^2(T^\infty). \]

The continuity of the functional \( \Lambda \) implies that \( L = L(H_\psi) \) and by (4.3) \( H_\psi \in BH \).

We have proved that the operator
\[ L : BH \to (H^2(T^\infty) \odot H^2(T^\infty))^* \], \( H_\psi \mapsto L(H_\psi) \]

is an isometric isomorphism and as a consequence \( BH \cong (H^2(T^\infty) \odot H^2(T^\infty))^* \) is a Banach space.

\[ \textbf{Theorem 4.0.3. (J. Ortega-Cerdà, K. Seip)} \] The Nehari theorem fails for multiplicative Hankel operators on \( H^2(T^\infty) \). There exists a bounded multiplicative Hankel operator on \( H^2(T^\infty) \) without bounded symbols.

\textbf{Proof.} We assume that every operator \( H_\psi \in BH \) admits symbol \( \phi \in L^\infty(T^\infty) \) and we consider the operator
\[ L : L^\infty(T^\infty) \to BH, \ \phi \mapsto H_\phi. \]

\( L \) is a bounded and surjective operator between Banach spaces. By the open mapping theorem there exists a constant \( \delta > 0 \) such that \( B_{BH}(0, \delta) \subset L(B_{L^\infty(T^\infty)}(0, 1)) \) or equivalently for every \( H_\psi \in BH \) there exists a bounded symbol \( \phi \in L^\infty(T^\infty) \) such that
\[ \| \phi \|_{L^\infty(T^\infty)} \leq C \| H_\psi \|, \] (4.4)

where \( C > 0 \) is an absolute constant.

Suppose \( d \) is an even natural number, we define the set
\[ I = \left\{ n \in \mathbb{N} : n = \prod_{k=1}^{\frac{d}{2}} q_k, q_k = p_{2k} \text{ or } q_k = p_{2k-1} \right\}. \]
We consider the analytic polynomial
\[
\psi(z) = \sum_{n \geq 1} \chi_I(n)z^n = \prod_{k=1}^d (z_{2k} + z_{2k-1}),
\]
where \(\chi_I\) is the characteristic function of \(I\). We will prove that \(\|H\psi\| = 2^d\) using the Schur test (Lemma 1.2.7.) for \(X = \mathbb{N}, \mu = \) the counting measure, \(H(n,m) = \chi_I(nm)\) and \(h(n) = 2^{-\omega(n)/2}\), where the function \(\omega(n)\) counts the prime factors of \(n \in \mathbb{N}\).

If \(n\) does not divide any integer in \(I\), then
\[
\sum_{m \geq 1} H(n,m)h(m)^2 = \sum_{m \geq 1} \chi_I(nm)2^{-\omega(m)/2} = 0.
\]
If \(n\) divides an integer in \(I\), then
\[
\sum_{m \geq 1} H(n,m)h(m)^2 = \sum_{m \geq 1} \chi_I(nm)2^{-\omega(m)/2}
= \sum_{nm \in I} 2^{-\omega(n)/2}
= 2^{d/2-\omega(n)/2}
= 2^d h(n)^2.
\]
This implies that \(\|H\psi\| \leq 2^d\) and trivially \(H_\psi(\psi) = 2^d, \|\psi\|_{H^2(\mathbb{T}^\infty)} = 2^d\). Thus,
\[
\|H\psi\| = 2^d. \tag{4.5}
\]
We observe that
\[
\|z_{2k} + z_{2k-1}\|_{L^1(\mathbb{T}^\infty)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{i\theta_1} + e^{i\theta_2}| d\theta_1 d\theta_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 + e^{i\theta}| d\theta = \frac{4}{\pi},
\]
\[
\|\psi\|_{L^1(\mathbb{T}^\infty)} = \left(\frac{4}{\pi}\right)^{\frac{d}{2}}. \tag{4.6}
\]
By (4.4), (4.5) and (4.6)
\[
2^d = \sum_{n \geq 1} \chi_I(n)^2 = \langle H_\psi(1), \psi \rangle = \langle \phi, \psi \rangle \leq \|\phi\|_{L^\infty(\mathbb{T}^\infty)} \|\psi\|_{L^1(\mathbb{T}^\infty)} \leq C 2^d \left(\frac{4}{\pi}\right)^{\frac{d}{2}}.
\]
So,
\[
C \geq \left(\frac{\pi}{2\sqrt{2}}\right)^{\frac{d}{2}} \to +\infty, \text{ as } d \to +\infty, \text{ contradiction.} \]
\[\blacksquare\]
**Theorem 4.0.4. (Brevig)**

Let \( \psi(z) = \frac{\sqrt{6}}{\pi} \sum_{k \geq 1} \frac{\psi_k}{k} \), where \( \psi_k(z) = \frac{h(k+1)}{2} \prod_{j = \frac{h+1}{2}+1}^{z_2j+z_2j-1} \sqrt{2} \), \( k \in \mathbb{N} \).

Then, the function \( \psi \in H^2(\mathbb{T}^\infty) \) induces a bounded multiplicative Hankel operator on \( H^2(\mathbb{T}^\infty) \) without bounded symbol.

**Proof.** Working as above we obtain that

\[
\| \psi_1 \|_{H^2(\mathbb{T}^\infty)} = \| \psi_k \|_{H^2(\mathbb{T}^\infty)} = \| \psi \|_{H^2(\mathbb{T}^\infty)} = 1
\]

\[
H_{\psi_k} = 1 \quad \text{and} \quad \| \psi_k \|_{H^1(\mathbb{T}^\infty)} = \left( \frac{4}{\pi \sqrt{2}} \right)^k, \quad k \in \mathbb{N}.
\]

Suppose \( f \) is a polynomial with norm that is equal to one. The set \( \{ \psi_k : k \in \mathbb{N} \} \) is orthonormal in \( H^2(\mathbb{T}^\infty) \) and as a consequence there exist complex numbers \( \lambda_0, \lambda_1, \lambda_2, \ldots \) and a unit vector \( g \in \text{span} \{ \psi_k : k \in \mathbb{N} \} \perp \) such that

\[
f = \sum_{k \geq 1} \lambda_k \psi_k + \lambda_0 g, \quad \sum_{n \geq 0} |\lambda_n|^2 = 1.
\]

We observe that

\[
H_{\psi}(f) = \frac{\sqrt{6}}{\pi} \sum_{k \geq 1} \frac{H_{\psi_k}(f)}{k}
\]

\[
= \frac{\sqrt{6}}{\pi} \sum_{k \geq 1} \frac{\lambda_k}{k} H_{\psi_k}(\psi_k) + \lambda_0 \frac{\sqrt{6}}{\pi} \sum_{k \geq 1} \frac{H_{\psi_k}(g)}{k}
\]

\[
= \frac{\sqrt{6}}{\pi} \sum_{k \geq 1} \frac{\lambda_k}{k} \frac{4}{\pi \sqrt{2}} \sum_{k \geq 1} \frac{H_{\psi_k}(g)}{k}.
\]

It is easy to prove that \( H_{\psi_k}(g) \perp H_{\psi_k'}(g), \ k \neq k' \) and by the Pythagorean theorem

\[
\| H_{\psi}(f) \|^2_{H^2(\mathbb{T}^\infty)} \leq \sum_{k \geq 1} |\lambda_k|^2 + |\lambda_0|^2 \frac{6}{\pi^2} \sum_{k \geq 1} \| H_{\psi_k}(g) \|^2_{H^2(\mathbb{T}^\infty)} \leq \sum_{n \geq 0} |\lambda_n|^2 = 1.
\]

This implies that \( H_{\psi} \) is a bounded multiplicative Hankel operator on \( H^2(\mathbb{T}^\infty) \) with norm that is equal to one (\( H_{\psi}(\psi) = 1 \)).

We assume that there exists a bounded symbol \( \phi \) for \( H_{\psi} \),

\[
\frac{\sqrt{6}}{\pi} \frac{1}{k} = \langle \psi_k, \psi \rangle = |\langle \psi_k, \phi \rangle| \leq \| \phi \|_{L^\infty(\mathbb{T}^\infty)} \| \psi_k \|_{H^1(\mathbb{T}^\infty)} = \| \phi \|_{L^\infty(\mathbb{T}^\infty)} \left( \frac{4}{\pi \sqrt{2}} \right)^k.
\]
Thus,
\[ \| \phi \|_{L^\infty(\mathbb{T}^\infty)} \geq \limsup_{k \to +\infty} \frac{\sqrt{6}}{\pi} \frac{1}{k} \left( \frac{\pi \sqrt{2}}{4} \right)^k = +\infty, \text{ contradiction.} \]

\[ \square \]

**Corollary 4.0.2.** \( H^1(\mathbb{T}^\infty) \) does not admit weak factorization, i.e.
\[ H^2(\mathbb{T}^\infty) \circ H^2(\mathbb{T}^\infty) \not\subseteq H^1(\mathbb{T}^\infty). \]

**Lemma 4.0.3.** We assume that every multiplicative Hankel operator that exists in the Schatten class \( S_p, \ p \geq 1 \) has a bounded symbol. Then, for every \( H_\psi \in S_p \) there exists a bounded symbol \( \phi \in L^\infty(\mathbb{T}^\infty) \) such that
\[ \| \phi \|_{L^\infty(\mathbb{T}^\infty)} \leq C_p \| H_\psi \|_{S_p}, \]
where \( C_p \) is a positive constant that depends only on \( p \).

**Proof.** We denote by \( S_p H \) the space of multiplicative Hankel operators on \( H^2(\mathbb{T}^\infty) \) that belong to the Schatten class \( S_p, \ p \geq 1 \). \( S_p H \) is a Banach space, since convergence in \( S_p H \subset S_p \) implies convergence in the Banach space \( BH \) (Corollary 1.1.3.).

Suppose \( M = L^\infty(\mathbb{T}^\infty) \cap H^2(\mathbb{T}^\infty) \), where \( H^2(\mathbb{T}^\infty) = \{ f \in L^2(\mathbb{T}^\infty) : \widehat{f}(a) = 0, \forall a \in \mathbb{N}_0^\infty \} \). \( M \) is a closed subspace of \( L^\infty(\mathbb{T}^\infty) \), we consider the quotient Banach space \( L^\infty(\mathbb{T}^\infty)/M \) and we define the operator
\[ L : S_p H \to L^\infty(\mathbb{T}^\infty)/M, \ H_\psi \mapsto \phi + M, \]
where \( \phi \) is a bounded symbol for \( H_\psi \). \( L \) is well defined, we will use the closed-graph theorem to prove that \( L \) is continuous. Let
\[ H_\psi_n \to H_\psi \text{ in } S_p \text{ and } \phi_n + M \to \phi + M \text{ in } L^\infty(\mathbb{T}^\infty)/M. \]
Then, there exists a sequence \( \{ g_n \}_{n \geq 1} \subset M \) such that
\[ \phi_n + g_n \to \phi \text{ in } L^\infty(\mathbb{T}^\infty). \]
We observe that
\[ \| H_\psi_n - H_\psi \|_{S_p} = \| H_{\phi_n + g_n} - H_\phi \|_{S_p} \leq \| \phi_n + g_n - \phi \|_{L^\infty(\mathbb{T}^\infty)} \to 0. \]
So, \( H_\psi = H_\phi, \ L(H_\psi) = \phi + M \) and as a consequence \( L \) is continuous with norm \( C_p > 0 \). This implies that for every \( H_\psi \in S_p H \) there exists a bounded symbol \( \phi \in L^\infty(\mathbb{T}^\infty) \) such that
\[ \| \phi \|_{L^\infty(\mathbb{T}^\infty)} \leq C_p \| H_\psi \|_{S_p}. \]
\[ \square \]
Lemma 4.0.4. Let $d, m \in \mathbb{N}$ and

$$
\psi_1(z) = \frac{z_1 + \ldots + z_d}{\sqrt{d}}, \quad \psi_2(z) = \frac{z_{d+1} + \ldots + z_{2d}}{\sqrt{d}}, \ldots, \quad \psi_m(z) = \frac{z_{(m-1)d+1} + \ldots + z_{md}}{\sqrt{d}},
$$

$$
\psi = \prod_{i=1}^{m} \psi_i.
$$

Then,

1. \hspace{1cm} \left\| H_{\psi} \right\|_{S_p} = \prod_{i=1}^{m} \left\| H_{\psi_i} \right\|_{S_p} = \left( \left\| H_{\psi_1} \right\|_{S_p} \right)^m = 2^m \, p, \quad p \geq 1.

2. \hspace{1cm} \lim_{d \to +\infty} \left\| \psi \right\|_{L^1(\mathbb{T}^\infty)} = \lim_{d \to +\infty} \left( \left\| \psi_1 \right\|_{L^1(\mathbb{T}^\infty)} \right)^m = \left( \frac{\sqrt{\pi}}{2} \right)^m.

Proof.

1. We want to find the canonical decomposition of $H_{\psi}$. First, we will consider the easy cases where $m = 1, 2$ and then we will work for an arbitrary $m \in \mathbb{N}$.

Let $f(z) = \sum_{n \geq 1} \hat{f}(n) z^n \in H^2(\mathbb{T}^\infty)$, then

$$
H_{\psi_i}(f) = \sum_{n \geq 1} \sum_{m \geq 1} \hat{\psi}_i(nm) \hat{f}(m) z^n
\quad = \sum_{k=1}^{d} \hat{\psi}_i(p(i-1)d+k) \hat{f}(1) z^{(i-1)d+k} + \sum_{k=1}^{d} \hat{\psi}_i(p(i-1)d+k) \hat{f}(p(i-1)d+k) \cdot 1
\quad = \langle f, 1 \rangle \psi_i + \langle f, \psi_i \rangle \cdot 1, \quad i = 1, \ldots, m.
$$

So, 1 is the unique singular value of $H_{\psi_i}$ with multiplicity 2. Thus,

$$
\left\| H_{\psi_i} \right\|_{S_p} = 2^{\frac{1}{p}}.
$$

Working similarly, we obtain that

$$
H_{\psi_i \psi_j}(f) = \langle f, 1 \rangle \psi_i \psi_j + \langle f, \psi_i \rangle \psi_j + \langle f, \psi_j \rangle \psi_i + \langle f, \psi_i \psi_j \rangle 1, \quad i \neq j.
$$

So, 1 is the unique singular value of $H_{\psi_i \psi_j}$ with multiplicity 4 and

$$
\left\| H_{\psi_i \psi_j} \right\|_{S_p} = 2^{\frac{2}{p}}.
$$

Now we consider the multiplicative Hankel operator $H_\psi = H_{\psi_1 \ldots \psi_m}$. We define the sets

$$
I_1 = \{p_1, \ldots, p_d\}, \quad I_2 = \{p_{d+1}, \ldots, p_{2d}\}, \ldots, \quad I_m = \{p_{(m-1)d+1}, \ldots, p_{md}\}
$$

and
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\[ I = I_1 \cdot I_2 \cdot \ldots \cdot I_m. \]

We observe that
\[ \hat{\psi}(n) = \left( \frac{1}{d} \right)^m, \; n \in I \quad \text{and} \quad \hat{\psi}(n) = 0, \; n \notin I. \]

\[
H_\psi(f) = \sum_{n \geq 1} \sum_{m \geq 1} \hat{\psi}(nm) \hat{f}(m) z^{\gamma(n)}
= \sum_{nm \in I} \frac{1}{d^m} \hat{f}(m) z^{\gamma(n)}
= \sum_{p_{ik} \in I_k} \sum_{\epsilon_k \in \{0, 1\}} \frac{1}{d^m} f \left( \prod_{k=1}^m p_{ik}^{\epsilon_k} \right) \prod_{k=1}^m z_i^{1-\epsilon_k}
= \sum_{\epsilon_k \in \{0, 1\}} \langle f, \psi_1^{\epsilon_1} \cdot \psi_2^{\epsilon_2} \cdot \ldots \cdot \psi_m^{\epsilon_m} \rangle \cdot 1.
\]

So, 1 is the unique singular value of \( H_\psi \) with multiplicity
\[ 2^m = \text{card} \{ (\epsilon_1, \ldots, \epsilon_m) : \epsilon_k \in \{0, 1\} \}. \]

Follows that
\[ \| H_\psi \|_{S_p} = 2^m. \]

2. Trivially
\[
\| \psi \|_{L^1(\mathbb{T}^\infty)} = \int \| \psi_1(z_1, \ldots, z_d) \cdot \psi_2(z_{d+1}, \ldots, z_{2d}) \cdot \ldots \cdot \psi_m(z_{(m-1)d+1}, \ldots, z_{md}) \| d\mu(z)
= \| \psi_1 \|_{L^1(\mathbb{T}^\infty)}^m.
\]
\[
\| \psi_1 \|_{L^1(\mathbb{T}^\infty)} = \int_{\mathbb{T}^d} \left| \frac{z_1 + \ldots + z_d}{\sqrt{d}} \right| d\mu(z)
= \left( \frac{1}{2\pi} \right)^d \int_0^{2\pi} \ldots \int_0^{2\pi} \left| 1 - \sum_{i=1}^d \cos(\theta_i) + i \sum_{i=1}^d \sin(\theta_i) \right| \prod_{i=1}^d d\theta_1 \cdot \ldots \cdot d\theta_d.
\]

We consider the sequence of Steinhaus random variables \( \{ e^{i\theta_k} \} \) and we observe that [30]

\[ \text{Var}(\cos(\theta_k)) = \text{Var}(\sin(\theta_k)) = \frac{1}{2} \quad \text{and} \quad \text{E}(\cos(\theta_k)) = \text{E}(\sin(\theta_k)) = 0. \]
By the Central Limit Theorem \[8\]

\[
\frac{1}{\sqrt{d}} \sum_{k=1}^{d} e^{i\theta_k} \sim Z_1 + iZ_2, \quad \text{as } d \to +\infty.
\]

\(Z_1, Z_2 \sim N(0, \frac{1}{2})\) with joint probability density function \(f(x, y) = \frac{1}{\pi} e^{-\left(x^2 + y^2\right)}\).

Follows that

\[
\lim_{d \to +\infty} \|\psi_1\|_{L^1(T^\infty)} = \mathbb{E}|Z_1 + iZ_2| = \mathbb{E}(Z_1^2 + Z_2^2)^{\frac{1}{2}}
\]

\[
= \frac{1}{\pi} \int \int \sqrt{x^2 + y^2} e^{-\left(x^2 + y^2\right)} dy dx
\]

\[
= \frac{\sqrt{\pi}}{2}.
\]

\[\Box\]

**Theorem 4.0.5. (Brevig, Perfekt)** For every \(p > p_0 = \left(1 - \frac{\log \pi}{\log 4}\right)^{-1}\) there exists a multiplicative Hankel operator \(H_\psi \in S_p\) which does not admit bounded symbol.

**Proof.** For \(p > p_0\), we assume that every \(H_\psi \in S_p\) admits bounded symbol. By the Lemma 4.0.3, for every \(H_\psi \in S_p\) there exists a bounded symbol \(\phi \in L^\infty(T^\infty)\) such that

\[
\|\phi\|_{L^\infty(T^\infty)} \leq C_p \|H_\psi\|_{S_p},
\]

where \(C_p\) is a positive constant that depends only on \(p\).

For \(m \in \mathbb{N}\), we consider the function \(\psi\) as in the Lemma 4.0.4. and we choose \(d \in \mathbb{N}\) sufficiently large such that

\[
\|H_\psi\|_{S_p} \|\psi\|_{L^1(T^\infty)} < 2^{\frac{1}{p_0}} \frac{\sqrt{\pi}}{2} = 1.
\]

There exists a bounded symbol \(\phi\) such that

\[
1 = \langle \psi, \psi \rangle = \langle H_\psi(1), \psi \rangle = \langle \phi, \psi \rangle \leq \|\phi\|_{L^\infty(T^\infty)} \|\psi\|_{L^1(T^\infty)} \leq C_p \|H_\psi\|_{S_p} \|\psi\|_{L^1(T^\infty)}.
\]

Thus,

\[
C_p \geq \left(\frac{1}{\|H_\psi\|_{S_p} \|\psi\|_{L^1(T^\infty)}}\right)^m \to +\infty \quad \text{as } m \to +\infty, \quad \text{contradiction}.
\]

\[\Box\]
Theorem 4.0.6. Let \( p > p_0 = \left( 1 - \frac{\log \frac{\pi}{2}}{\log 4} \right)^{-1} \), \( \frac{\sqrt{\pi}}{2} < a < 2^{-\frac{1}{p}} \), \( d = d(a) \in \mathbb{N} \) sufficiently large and

\[
\psi = \sum_{k \geq 1} a^k \psi_k,
\]

where

\[
\psi_{1,1} = \frac{z_1 + z_2 + \ldots + z_d}{\sqrt{d}}, \quad \psi_1 = \psi_{1,1},
\]

\[
\psi_{2,1} = \frac{z_1 + z_2 + \ldots + z_{2d}}{\sqrt{d}}, \quad \psi_{2,2} = \frac{z_2 + z_2 + \ldots + z_{2d}}{\sqrt{d}}, \quad \psi_2 = \psi_{2,1} \cdot \psi_{2,2},
\]

\[
\vdots
\]

\[
\psi_k,1 = \frac{z_{d(k-1)} + z_{d(k-1)} + \ldots + z_{d(k-1)+1}}{\sqrt{d}}, \quad \ldots, \quad \psi_k,\ldots, \psi_k,k = \frac{z_{d(k+1)} + z_{d(k+1)} + \ldots + z_{d(k+1)}}{\sqrt{d}},
\]

\[
\psi_k = \prod_{i=1}^{k} \psi_{k,i}, \quad k \in \mathbb{N}.
\]

Then, \( H_\psi \) is a multiplicative Hankel operator, that belongs to the Schatten class \( S_p \), without bounded symbol.

Proof. The sequence \( \{\psi_k\}_{k \geq 1} \subset H^2(\mathbb{T}^\infty) \) is orthonormal and by Parseval’s formula we obtain that \( \psi \in H^2(\mathbb{T}^\infty) \)

\[
\|\psi\|_{H^2(\mathbb{T}^\infty)} = \left( \sum_{k \geq 1} a^{2k} \right)^{\frac{1}{2}} = \frac{a}{\sqrt{1-a^2}}.
\]

We observe that the set \( \left\{ \prod_{i=1}^{k} \psi_{k,i}^{\epsilon_i} : k \in \mathbb{N}, i = 1, \ldots, k \text{ and } \epsilon_i \in \{0,1\} \right\} \) is orthonormal and working as in the Lemma 4.0.4.

\[
H_\psi(f) = \sum_{k \geq 1} \sum_{\epsilon_i \in \{0,1\}} a^k \langle f, \prod_{i=1}^{k} \psi_{k,i}^{\epsilon_i} \prod_{i=1}^{k} \psi_{k,i}^{1-\epsilon_i} \rangle
\]

\[
= \sum_{k \geq 1} a^k \left( \langle f, 1 \rangle \psi_{k,1} \cdot \ldots \cdot \psi_{k,k} + \langle f, \psi_{k,1} \rangle \psi_{k,2} \cdot \ldots \cdot \psi_{k,k} + \ldots + \langle f, \psi_{k,1} \cdot \ldots \cdot \psi_{k,k} \rangle \cdot 1 \right),
\]

\[
\|H_\psi(f) - \langle f, \psi \rangle \cdot 1 - \langle f, 1 \rangle \psi\|_{S_p}^p = \sum_{k \geq 1} a^{pk} \left( 2^k - 2 \right) < +\infty.
\]

Follows that \( H_\psi \in S_p \).

We have already proved (Lemma 4.0.4.) that for \( d \) sufficiently large

\[
\left\| \frac{z_1 + z_2 + \ldots + z_d}{\sqrt{d}} \right\|_{L^1(\mathbb{T}^\infty)} \sim \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \|\psi_k\|_{L^1(\mathbb{T}^\infty)} \sim \left( \frac{\sqrt{\pi}}{2} \right)^k.
\]
We can choose \( d \in \mathbb{N} \) such that
\[
\frac{a}{\| \psi_1 \|_{L^1(T^\infty)}} > 1.
\]

We assume that there exists a bounded symbol \( \phi \) for the operator \( H_\psi \).

\[
a^k = \langle \psi, \psi_k \rangle = \langle \phi, \psi_k \rangle \leq \| \phi \|_{L^\infty(T^\infty)} \| \psi_k \|_{L^1(T^\infty)}.
\]

Thus,
\[
\| \phi \|_{L^\infty(T^\infty)} \geq \limsup_{k \to +\infty} \left( \frac{a}{\| \psi_1 \|_{L^1(T^\infty)}} \right)^k = +\infty, \text{ contradiction.}
\]

\[\blacksquare\]
Bibliography


