On the role and the implications of large-scale peculiar motions for the deceleration parameter of the universe

Miltiadis I. Kadiltzoglou

Department of Physics, Aristotle University of Thessaloniki
Thessaloniki 54124, Greece

Submitted for the degree of Doctor of Philosophy
November 2021
No real observer in the universe follows the smooth Hubble expansion, but we all move relative to it. Our Milky Way and the Local Group of galaxies, in particular, drift at a speed of around 600 km/sec. Also, a great number of recent surveys have repeatedly confirmed the presence of large-scale peculiar motions, the so-called “bulk flows”. Despite this, peculiar motions are typically bypassed in most theoretical studies and, in the few studies they are included, the analysis is almost always Newtonian and takes the viewpoint of the idealised Hubble-flow observers, rather than that of their real bulk-flow counterparts. As a result, the full implications of our motion relative to the smooth universal expansion may not have been fully accounted for. However, relative-motion effects have long been known to interfere with the way the associated observers interpret their cosmos. In fact, the history of astronomy is rife with examples where relative motions have led to a gross misinterpretation of reality. This Thesis aims to provide a fully relativistic treatment of bulk peculiar flows and to investigate their implications for the way we understand the mean kinematics of the universe we live in and more specifically its acceleration/deceleration rate. The latter is monitored by the deceleration parameter, which traditionally is positive in a decelerated cosmos and takes negative values in an accelerating one. Introducing a “tilted” cosmological model, we allow for two families of observers. The first follows the smooth Hubble flow, which also defines the reference frame of the universe, while the second group of observers lives in typical galaxies, like our Milky Way, that drift relative to it. Assuming a perturbed Friedmann universe filled with pressure-free dust, we show that the deceleration parameters measured by these two observer groups differ and that their difference is entirely due to relative-motion effects. In addition, using linear cosmological perturbations theory, we find that observers residing inside slightly contracting bulk flows may assign negative values to their locally measured deceleration parameter in a universe that is globally decelerating. Although this is an apparent local effect that is triggered solely by the observers’ peculiar motion, the affected scales are typically large enough (between few hundred and several hundred Mpc) to create the false impression that the whole universe has recently entered a phase of accelerated expansion. Indications that the scenario outlined above may be true and that the inferred recent universal acceleration may be a mere illusion and an artefact of our peculiar motion relative should be sought in the data. These should contain, among others, the trademark signature of relative motion, namely an apparent (Doppler-like) dipolar anisotropy triggered by the observers’ peculiar flow. In other words, in the data, the universe should appear to accelerate faster in one direction and equally slower in the opposite. Intriguingly, over the last ten years or so, there have been reports in the literature that such a dipole axis, may actually reside in the supernovae data. Put another way, our universe may indeed appear to accelerate faster towards one direction in the sky and equally slower along the antipodal.
Acknowledgments

This work would not have been possible without the constant support, guidance, encouragement and assistance from my advisor Prof. Christos Tsagas. His levels of patience, knowledge and ingenuity is something I will always keep aspiring to.

I would also like to thank my former teachers and all the staff at the Laboratory of Astronomy in Thessaloniki. Special thanks also to the Lab secretary Mrs. Lemonia Touloumi. Her smile, politeness and help made the Observatory an ideal working environment.

Last, but not least, my warm and heartfelt thanks go to my family for their tremendous support, hope and continuous encouragement they had given to me all these years. Immense gratitude, as always, to my wife Christina for her patience and support, as well as for the birth of our daughter during the final stages of my thesis.

The later phases of this work were pursued as part of the program “Tilted Cosmology”, which is financially supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.), under the “First Call for H.F.R.I. Research Projects to support Faculty members and Researchers and the procurement of high-cost research equipment grant” (Project Number: 789).
1 Introduction

The current cosmological paradigm, treats the universe we live in as nearly Friedmann-Robertson-Walker (FRW) spacetime, with almost homogeneous and isotropic spatial sections on sufficiently large scales, which expand at an accelerated space. This universal acceleration, which according to the $\Lambda$CDM paradigm has been a relatively recent phenomenological addition to the kinematics of the cosmos, has been based on observations of distant supernovae reported at the end of the last millennium [1,2]. Given that all the known forms of matter decelerate the expansion, the answer was sought to exotic sources with negative gravitational mass/energy. This is the familiar “dark energy”, which tends to repulse (instead of attract) gravitationally. Alternatively, one can appeal to the presence of a (positive) cosmological constant capable of dominating over the rest of the material content of the universe and thus drive the expansion into an accelerated phase. Although other explanations to the supernovae observations, such as modifying general relativity, abandoning the FRW models, or introducing extra dimensions, have been proposed in the literature, dark energy and the cosmological constant are currently the most popular among the cosmological community (e.g. see [3] for a review).

Nevertheless, neither dark energy nor the cosmological constant are problem-free options (e.g. see [4,5]). The former is an exotic form of matter, which though it has never been detected (so far at least), it happens to have all the desired properties to explain the observations. Among others, dark energy does not interact with ordinary matter, but only gravitationally, it is uniformly distributed in space and evolves in time in a (rather fine-tuned) way that allows it to dominate the kinematics of the universe now and not at a much earlier, or at a much later, time. This leads to the so-called “coincidence problem”. The cosmological constant, as well, is a rather ad hoc addition to the dynamics of our cosmos, having a magnitude that is some one hundred and twenty (120) orders of magnitude smaller that its theoretically predicted value. In particular, if the cosmological constant is to dominate the kinematics of the universe today, its value must be essentially identical to that of the Hubble parameter today. Such a coincidence between the value of a fundamental (universal) constant and that of a dynamically evolving parameter is also quite difficult to explain. The rest of the most common alternatives also suffer from analogous/similar problems.

One of the reasons, perhaps the main one, that have led to the dark-energy/cosmological-constant paradigm is the widespread perception that accelerated expansion is impossible within standard general relativity, as long as conventional matter fills the universe. This, however, is a working hypothesis rather than a hard theoretical fact. General relativity has not been exhausted yet. There are still unexplored aspects of Einstein’s theory that could, in principle at least, drastically change our current views of the universe we live in. The aim of the present thesis is to explore one of these aspects and its potential cosmological implications.

Peculiar motions are commonplace in our universe. Our galaxy and the Local Group of galaxies, for example, “drift” at approximately 600 km/sec relative to the mean universal expansion (e.g. see [6]). The latter sets the idealised reference frame of the universe, with respect to which
peculiar velocities can be defined and measured. Technically, the Hubble frame is identified with that of the Cosmic-Microwave-Background (CMB) radiation, which in turn is defined as the coordinate system where the CMB dipole vanishes. In addition to the local motion of our Milky Way and its neighbouring galaxies, many surveys have repeatedly confirmed the presence of large-scale peculiar motions (e.g. see [7–15] for a representative though incomplete list). These are commonly referred to as “bulk flows” and consist of large domains of the observable universe moving coherently towards a given direction in the sky. The typical sizes of these bulk flows are few hundred Mpc and their typical velocities few hundred km/sec. Larger and faster peculiar motions, namely the controversial “dark flows”, have also been reported in the literature [16–18]. All these drift motions are believed to be a relatively recent addition to the kinematics of our cosmos and the unavoidable consequence of the ongoing structure formation process (e.g. see [19,20]). Overall, it seems quite reasonable to argue that no real observer in the universe follows the smooth Hubble expansion, but we all have some finite peculiar velocity relative to it. Despite that, peculiar velocities have been bypassed in almost all of the available theoretical cosmological studies. In addition, when peculiar motions are included, the analysis is usually Newtonian and it is done in the rest-frame of the idealised (Hubble-flow) observers, instead of the coordinate system of their real (bulk-flow) counterparts.

Relative motions have long been known to interfere with the way the associated observers interpret and understand their surroundings. In fact, the history of astronomy contains a number of examples where relative-motion effects have led to a serious misinterpretation of reality. This is particularly true in relativity, since in Einstein’s theory moving observers measure their own time and space and generally disagree on their temporal and spatial measurements. On these grounds, it is conceivable that the aforementioned idealised Hubble-flow observers may “see” a different universe than the one “experienced” by their real counterparts, namely by those living in typical galaxies like our Milky Way. Our aim is to explore this possibility and, more specifically, establish the extent relative-motion effects can interfere with the way the associated observers interpret the mean kinematics of their host universe.

Here, we focus and compare the deceleration parameters, as measured by the observers moving along with the smooth Hubble flow and by those living in typical galaxies drifting relative to the mean universal expansion. In so doing, we employ general relativity and more specifically we use the 1+3 covariant approach to cosmology (see [21,22] for recent extended reviews). The covariant formalism is applied to a perturbed, “tilted”, almost-FRW universe filled with ordinary low-energy “dust”. Tilted cosmological models allow for two (at least) families of observers moving relative to each other (e.g. see [23,24]). Typically, one group of observers is identified with the smooth Hubble expansion, so that their coordinate system defines the reference frame of the universe (i.e. the CMB frame). The other group, on the other hand, resides in galaxies like our Milky way, which have some finite peculiar velocity with respect to the Hubble flow. Even when the peculiar velocities involved are non-relativistic, as it happens to be in our case, the expansion rates (that is the Hubble parameters) measured by the aforementioned two groups of observers differ. Moreover, the difference is entirely due to relative-motion effects. Here, we find that the same is also true for the deceleration parameters measured by the Hubble-flow and by the bulk-flow observers. Furthermore, using linear
relativistic cosmological perturbation theory, we establish that the difference between the de- 
celeration parameters measured by these two observer groups also depends on the specifics of 
the bulk peculiar flow.

Clearly the speed of the bulk flow is a key factor in determining the overall impact of the 
relative-motion effects. There is an additional scale-dependence, however, which ensures that 
the effects increase as we move down to progressively smaller scales, namely closer to the 
observer. On the other hand, the impact of relative motion is found to drop as one moves 
away from the observer. In practice, this means that on relatively small wavelengths the 
deceleration parameters measured in the Hubble and in the “tilted” (i.e. the bulk flow) frame 
may differ considerably, whereas they are essentially the same on sufficiently long wavelengths. 
This last result agrees with the general perception that the effects of peculiar motions fade 
away on large enough scales.

At this point, we should emphasise that the aforementioned scale dependence of the relative- 
motion effects on the deceleration parameter is a purely general relativistic effect, which cannot 
be naturally reproduced in Newtonian physics. The reason is the fundamentally different way 
the two theories treat both the gravitational field and its sources. According to Newton, 
gravity is a force triggered by spatial variations in the gravitational potential. Moreover, only 
the density of the matter contributes to the gravitational field via Poisson’s equation. In 
relativity, on the other hand, gravity is the manifestation of spacetime curvature. The latter 
is caused by the presence of matter, which contributes to the gravitational field by means 
the local energy-momentum tensor. Crucially, in relativity, it is not only the matter density 
that “adds” to the gravitational field. The pressure (both isotropic and anisotropic) and any 
energy flux that may exists contribute as well. When dealing with bulk peculiar motions, 
it is this additional energy-flux input to the local gravitational field that separates the two 
theories [25,26]. As a result, the Newtonian energy and momentum conservation laws differ 
form their relativistic counterparts. These differences then feed into the formulae monitoring 
the evolution of peculiar-velocity perturbations and eventually change the relative-motion 
effects on the local kinematics and on the locally measured deceleration parameter (see [27] 
for a comparison and further discussion).

One more crucial finding is that the impact of relative motion on the deceleration parameter 
also depends on whether the bulk flow, where the tilted observers reside in, is locally expanding 
or contracting. More specifically, the deceleration parameter measured by observers inside 
locally expanding peculiar motions is found to be larger than the one measured in the Hubble 
frame. When dealing with (locally) contracting bulk flows, on the other hand, the effect is 
reversed. There, the tilted observers assign smaller values to their deceleration parameter. 
Nevertheless, in either case, the effect is not “real”, but a local artefact of the observers 
motion relative to the mean universal expansion.

Of the two cases outlined above, the second is the most intriguing, since it opens the theoretical 
possibility that the bulk-flow observers may assign negative values to their deceleration 
parameter, while at the same time the latter remains positive in the frame of their Hubble- 
flow counterparts [28–30]. If so, the tilted observers will locally “experience” the illusion of
accelerated expansion in a universe that is still globally decelerating. This happens on scales that are sufficiently “contaminated” by the relative-motion effects. In fact, there is a characteristic length, that we will from now on refer to as the “transition scale”, below which linear peculiar-velocity perturbations dominate over the background Hubble expansion and therefore dictate the local kinematics [31]. Physically, this transition scale is closely analogous to the familiar “Jeans length” (e.g. see [22,20,21]). The latter is also a result of linear perturbation theory and marks the threshold below which pressure-gradient perturbations take over the background gravity and thus determine the linear evolution of density perturbations.

Turning to the recent bulk flow surveys and using their peculiar-velocity measurements, we find that the transition scale is generally larger than the reported bulk-flow scale and typically varies from few hundred to several hundred Mpc. Therefore, regions as large as one tenth of the observable universe (perhaps even larger) around any typical observer in the cosmos can be heavily contaminated by relative-motion effects. On these grounds, theoretical conclusions based on data collected locally, namely from regions smaller or around the observer’s transition scale, should not be readily applied globally. For instance, when the bulk flow is locally contracting, data collected within or close to the transition scale, namely to the threshold where the deceleration parameter changes sign from positive to negative, could create the false impression that the whole universe recently entered a phase of accelerated expansion. In such a case, an unsuspecting observer can easily misinterpret a local (apparent) change in the sign of the deceleration parameter, as a recent (real) change in the global kinematics of the host universe. Overall, in order to make sure that their data have not been seriously contaminated by relative-motion effects, the observers should not “look” locally but at considerably high redshifts.

Is it then conceivable that the recent accelerated expansion of the universe may be a illusion and an mere artefact of our peculiar motion relative to the smooth Hubble expansion? The answer should be sought in the data. The latter should contain the imprint of the relative-motion effects outlined above. One on them is that the deceleration parameter should take progressively less negative values as one moves away from the observer, cross the zero threshold at the corresponding transition scale/redshift and become positive after that, eventually approaching its background value on large enough scales/redshifts. The detailed profile of the redshift distribution of the deceleration parameter depends on the specifics of the bulk flow the observer resides in. Nevertheless, the theoretical result obtained in the simplest case agrees qualitatively with the current data, claiming that the accelerated expansion of the universe turns into deceleration at high redshifts. In addition to the aforementioned redshift-distribution of the deceleration parameter, the data should also contain the “trademark” signature of relative motion, namely an apparent (Doppler like) dipolar anisotropy in the sky-distribution of the deceleration parameter. More specifically, the latter should appear to be more negative towards a given direction in the sky and equally less negative in the opposite. Put another way, the universe should appear to accelerate faster towards one point in the celestial sphere and equally slower in the antipodal. Moreover, the associated dipole axis should not lie very far form the one seen in the CMB spectrum, assuming that they are both the result of our peculiar motion relative to smooth universal expansion.
Over the last decade or so, there have been a number of surveys reporting that a dipolar anisotropy, fairly close to the CMB-axis, may actually reside in the supernovae data [32–37]. In other words, our universe may indeed seem to accelerate faster in one direction and equally slower in the opposite. Nevertheless, it was only recently that the aforementioned dipole was attributed to relative motion effects. This was done in the study of [38], which reconstructed the JLA data back to their original form and then looked for a dipolar anisotropy in them. The same study also found that the presence of the dipole reduced the prominence of the monopole. Put another way, the negative value of the deceleration parameter became less significant, thus increasing the possibility the recently accelerated expansion of the universe to be a relative-motion artefact. Future observations and more refined data should strengthen, or weaken, the observational support for this scenario.

As a closing comment we should state that, if nothing else, this work tries to draw attention to the potentially pivotal implications of large-scale peculiar motions for the kinematics of our universe. Relativistic structure formation studies (either covariant or metric-based) are abundant in the literature. At the linear perturbative level, these treatments are also known to agree with each other (e.g. see [39] for a comparison). Nevertheless, to the best of our knowledge, relatively few studies focus on the implications of the bulk peculiar flows and the reasons vary. For example, some treatments are performed in the so-called comoving gauge, where peculiar velocities are zero by default (e.g. see [40]). Others, although allowing for multi-fluid systems, are done in the Landau-Lifshitz (or energy) frame, where the total flux of the species vanishes (e.g. see [22,21] and also [41]). There are also the so-called “quasi-Newtonian” treatments. These start relativistic but along the way introduce an approximation, in the form of an effective gravitational potential analogous to that of the Newtonian studies, to account for the effects of peculiar motions (e.g. see [42]). In all these cases, the role of the bulk peculiar flows is either bypassed or downgraded and, as a result, their full implications remain largely unaccounted for.

The present Thesis starts with a brief introduction to the 1+3 covariant approach to cosmology, which provides the mathematical formalism adopted in our study. This take place in sections § 2 and § 3, before proceeding to the presentation of the idealised Friedmann models and to a discussion of their more realistic perturbed counterparts (see § 4 and § 5 respectively). An introduction to the tilted cosmological models is given in § 6 and the 1+3 formalism is applied to almost-FRW universes in § 7. The impact of relative motion on the deceleration parameter is analysed in § 8. There, we provide estimates for its value, measured locally by the relatively moving observers, as well as for the size of the associated transition length. All our numerical results are based on the peculiar velocities reported by recent bulk-flow surveys. Finally, we compare the relativistic to the Newtonian study in sections § 9 and § 10. There, we find that, within the framework of Newton’s theory, relative-motion effects have an entirely negligible impact on the deceleration parameters measured by the Hubble-flow and by the bulk-flow observers. We also identify and explain that the underlying reason the two theories lead to so different results and conclusions is the fundamentally different way they treat the gravitational field and its sources.
Before closing our introduction, we should also mention that the research work presented in the current Thesis is based on the following three publications:

2 The 1+3 covariant description

The covariant approach to fluid dynamics was first introduced in the 1950s through the work of Heckmann, Schücking, and Raychaudhuri [43,44]. The formalism was initially employed within the realm of the Newtonian theory, before applied to General Relativity originally by Ehlers and Ellis [45–47] and ultimately by a host of authors on a wide range of applications (e.g. see [48–65] for a representative list). The formalism uses the kinematic quantities, the energy-momentum tensor of the matter and the gravito-electromagnetic parts of the Weyl tensor, instead of the metric, which in itself does not provide a covariant description. Then, the key evolution and constraint equations emerge by means of the Ricci and Bianchi identities.

2.1 Local spacetime splitting

Consider a spacetime with Lorentzian metric $g_{ab}$ of signature (−, +, +, +) and introduce a group of observers “living” along timelike worldlines tangent to the 4-velocity field\(^1\)

$$u^a = \frac{dx^a}{d\tau},$$ (2.1.1)

where $u_a u^a = -1$ and $\tau$ measures the observers’ proper time. The above defined 4-velocity field introduces a local 1+3 ‘threading’ of the spacetime into time and space. More specifically, the 4-vector $u_a$ defines the time direction and the tensor $h_{ab} = g_{ab} + u_a u_b$ projects orthogonal to $u_a$ into the observers’ 3-dimensional instantaneous rest-space. When there is no vorticity (see § 3.1 below), the 4-velocity field is hypersurface-orthogonal and $h_{ab}$ acts as the metric of the aforementioned spatial sections.

Employing $u_a$ and $h_{ab}$ one can decompose every spacetime variable, operator and equation into their irreducible timelike and spacelike components. These two fields are also used to define the covariant temporal and spatial derivatives of any given tensor field $S_{abcd...}$ by means of

$$\dot{S}_{abcd...} = u^e \nabla_e S_{abcd...} \quad \text{and} \quad D_e S_{abcd...} = h_{ef} h_{ag} h_{bd} h_{cf} \ldots \nabla_b S_{ef...qr...}$$, (2.1.2)

respectively. For instance, projecting the 4-D Levi-Civita tensor ($\eta_{abcd}$, with $\eta_{abcd} = \eta_{(abcd)}$, $\eta_{0123} = 1$, $\eta_{abcd} \eta^{efpq} = -4! \delta^{[a}_{e} \delta^{b}_{f} \delta^{c}_{p} \delta^{d}_{q}]$ and $\nabla_e \eta_{abcd} = 0$) along $u_a$ leads to its 3-D counterpart, namely

$$\varepsilon_{abc} = \eta_{abcd} u^d.$$ (2.1.3)

It therefore follows that $\varepsilon_{abc} u^a = 0$,

$$\eta_{abcd} = 2 u_{[a} \varepsilon_{b] cd} - 2 \varepsilon_{abc} u[d] \quad \text{and} \quad \varepsilon_{abc} \varepsilon^{def} = 3! h_{[a}^d h_{b}^e h_{c]}^f.$$ (2.1.4)

\(^1\) Latin indices take values from 0 to 3, while Greek ones run from 1 to 3. Note that we use geometrised units with $c = 1 = 8\pi G$, which means that the geometrical variables have dimensions which are integer powers of length.
Also note that $D_c h_{ab} = 0 = D_d \varepsilon_{abc}$ by construction, while $\dot{h}_{ab} = 2u_1 A_b$ and $\dot{\varepsilon}_{abc} = 3u_1 \varepsilon_{bc} dA^4$ (with $A_a = \dot{u}_a$ representing the 4-acceleration – see § 3.1 below).

2.2 The gravitational field

General relativistic advocates a geometrical interpretation of gravity, which is no longer a force by the manifestation of spacetime curvature and the departure from Euclidean geometry. In Einstein’s theory matter induces spacetime curvature and the latter dictates the motion of the matter. This interplay is monitored by the Einstein field equations,

\[ G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = T_{ab} - \Lambda g_{ab}, \tag{2.2.1} \]

where $G_{ab}$ is the Einstein tensor, $R_{ab} = R_{acbc}$ is the spacetime Ricci tensor (with trace $R = R^{a}_{a} = g^{ab} R_{ab}$), $T_{ab}$ is the energy-momentum tensor of the matter and $\Lambda$ is the cosmological constant. Then, the twice-contracted Bianchi identities guarantee that $\nabla_b G_{ab} = 0$ and total energy-momentum conservation (see § 3.2 later).

The Ricci tensor describes the local gravitational field at each spacetime event due to the presence of matter there. The non-local, that is the long-range gravitational field, propagating by means of gravitational waves and tidal forces, is encoded in the Weyl (conformal curvature) tensor $C_{abcd}$. The local and the non-local components of the gravitational field satisfy the relation

\[ R_{abcd} = C_{abcd} + \frac{1}{2} \left( g_{ac} R_{bd} + g_{bd} R_{ac} - g_{bc} R_{ad} - g_{ad} R_{bc} \right) - \frac{1}{6} R \left( g_{ac} g_{bd} - g_{ad} g_{bc} \right), \tag{2.2.2} \]

with $R_{abcd}$ representing the Riemann (curvature) tensor. The latter describes the total gravitational field, it has the symmetries

\[ R_{abcd} = R_{cdab}, \quad R_{abcd} = -R_{bacd} = -R_{abdc} \quad \text{and} \quad R_{a[bc]d} = 0 \tag{2.2.3} \]

and also satisfies the Bianchi identities

\[ \nabla_\epsilon R_{ab\epsilon} \rho d = 0. \tag{2.2.4} \]

The Weyl tensor has all the symmetries of the Riemann tensor and is also trace-free. Although $C_{abcd}$ represents the part of the curvature that is not determined by the local matter, it is not unconstrained. In fact, contracting the Bianchi identities (see (2.2.4) above) once and using decomposition (2.2.2), gives

\[ \nabla^d C_{abcd} = \nabla_{[b} R_{a]c} + \frac{1}{6} g_{c[b} \nabla_{a]} R, \tag{2.2.5} \]

which act as the field equations of the Weyl tensor [21,22]. Relative to the $u_a$ field, the
conformal curvature tensor decomposes into its irreducible parts (e.g. see \[66,67\])

\[
E_{ab} = C_{abcd}u^c u^d \quad \text{and} \quad H_{ab} = \frac{1}{2} \varepsilon_a^c d_C C_{cdbe} u^e,
\]
respectively known as the electric and the magnetic Weyl tensors. These are both symmetric \((E_{ab} = E_{ba} \text{ and } H_{ab} = H_{ba})\), traceless \((E_a^a = 0 = H_a^a)\) and spacelike \((E_{ab} u^b = 0 = H_{ab} u^b)\). In addition,

\[
C_{abcd} = (g_{abqp} g_{cdsr} - \eta_{abqp} \eta_{cdsr}) u^q u^s E^{pr} - (\eta_{abqp} g_{cdsr} + g_{abqp} \eta_{cdsr}) u^q u^s H^{pr},
\]
where \(g_{abcd} = g_{ac} g_{bd} - g_{ad} g_{bc}\) \[21,22\]. Alternatively, we may write

\[
C_{ab}^{cd} = 4 \left( u[a u^c + h_a^c] E_b^d \right) + 2 \varepsilon_{abe} u^c H^{de} + 2 u[a H_b^e] \varepsilon^{cde}.
\]
Note that the electric Weyl component generalises the Newtonian tidal tensor, but \(H_{ab}\) has no Newtonian analogue. This relates the magnetic part of the Weyl field closer with gravitational waves, though both tensors are necessary to guarantee a nonzero super-energy flux vector \((P_a = \varepsilon_{abc} E^{bd} H_c^d)\) and the propagation of gravitational radiation.

2.3 Matter fields

Relative to observers moving with 4-velocity \(u_a\), the energy-momentum tensor of a general (imperfect) fluid decomposes as \(^2\)

\[
T_{ab} = \rho u_a u_b + p h_{ab} + 2 q(a u_b) + \pi_{ab},
\]
where \(\rho = T_{ab} u^a u^b\) is the energy density of the matter, \(p = T_{ab} h^{ab}/3\) is its isotropic pressure, \(q_a = -h_a^b T_{bc} u^c\) is the energy flux and \(\pi_{ab} = h_{(a}^c h_{b)}^d T_{cd}\) is the viscosity. \(^3\)

The 4-velocity filed \((u_a)\) is arbitrary and a velocity (Lorentz) boost introduces changes in the dynamical quantities (see \[68\] and also Appendix A.1 for more details). When dealing with a perfect fluid, however, there is a uniquely defined 4-velocity, relative to which \(q_a, \pi_{ab}\) are identically zero. In such a case,

\[
T_{ab} = \rho u_a u_b + p h_{ab}.
\]

\(^2\) In a multi-component medium, or when there are observers with peculiar velocities, we need to allow for the different 4-velocities of the each matter component, or of every observer (see § 6).

\(^3\) Angled brackets indicate the symmetric, traceless component of spatially projected second-rank tensors and also the projected part of vectors, namely

\[
S_{(ab)} = h_{(a}^c h_{b)}^d S_{cd} = h_{(a}^c h_{b)}^d S_{cd} - \frac{1}{3} h^{cd} S_{cd} h_{ab} \quad \text{and} \quad V_{(a)} = h_{a}^{b} V_{b},
\]
by construction (since \(S_{(ab)} h^{ab} = 0\)). We refer the reader to \[21,22\] for further discussion and additional technical details on the issue of covariant decomposition.
If we additionally set \( p = 0 \), we arrive at the simplest form of pressure-free matter, commonly known as ‘dust’, which typically refers to (post-recombination) baryonic species or/and Cold Dark Matter (CDM). Otherwise, we need to specify \( p \) as a function of \( \rho \) and possibly of other thermodynamic variables. Barotropic media, for example, have \( p = p(\rho) \), though the equation of state generally takes the form \( p = p(\rho, s) \), with \( s \) representing the specific entropy.

Taking the trace of (2.2.1) we find that \( R = 4\Lambda - T \), with \( T = T_a^a \). Then, Einstein’s field equations recast into

\[
R_{ab} = T_{ab} - \frac{1}{2} T g_{ab} + \Lambda g_{ab}. \tag{2.3.4}
\]

By successively contracting the above, while assuming that \( T_{ab} \) is given by (2.3.1), we arrive to the following set of algebraic relations

\[
R_{ab} u^a u^b = \frac{1}{2} (\rho + 3p) - \Lambda, \tag{2.3.5}
\]

\[
h^b_a R_{bc} u^c = -q_a, \tag{2.3.6}
\]

\[
h^a_e h^d_b R_{cd} = \frac{1}{2} (\rho - p) h_{ab} + \Lambda h_{ab} + \pi_{ab}, \tag{2.3.7}
\]

which will prove useful later.
3 Covariant relativistic cosmology

In cosmology, there are various physical alternatives for the fundamental 4-velocity field \( u_a \) that defines the \( 1 + 3 \) splitting of the spacetime. The most common choice identifies \( u_a \) with the frame where the dipole of the CMB anisotropy vanishes.

3.1 Kinematics

The motion of the observers is fully determined by its irreducible kinematical variables. These emerge by decomposing the 4-velocity gradient according to

\[
\nabla_b u_a = \sigma_{ab} + \omega_{ab} + \frac{1}{3} \Theta h_{ab} - A_a u_b ,
\]

with \( \sigma_{ab} = D_b u_a \), \( \omega_{ab} = D_b u_a \), \( \Theta = \nabla^a u_a = D^a u_a \) and \( A_a = \dot{u}_a = u_b \nabla_b u_a \). These are respectively the shear and the vorticity tensors, the volume expansion/contraction scalar, and the 4-acceleration vector. The latter implies the presence of non-gravitational forces and vanishes when matter moves along (timelike) geodesics under gravity alone. By construction, the shear, the vorticity and the 4-acceleration are all spacelike, with \( \sigma_{ab} u^a = 0 = \omega_{ab} u^a = A_a u^a \). The volume scalar monitors the average separation between two neighbouring observers and also introduces a representative length scale (the cosmological scale factor \( a \)), defined by \( \dot{a}/a = \Theta/3 \). Nonzero vorticity changes the orientation of the fluid element, but not its volume or shape. Also, the rotation axis is determined by the vorticity vector \( \omega_a = \varepsilon_{abc} \omega^b \omega^c / 2 \) (so that \( \omega_{ab} = \varepsilon_{abc} \omega^c \)). Finally, the effect of the shear is to alter the shape of a given fluid element, while leaving its volume unchanged.

The full kinematics of the \( u_a \)-field is determined by three evolution formulae and by three constraint equations. Both sets are purely geometrical in origin, essentially independent of the Einstein equations and emerge after applying the Ricci identities

\[
2 \nabla_{[a} \nabla_{b]} u_c = R_{abcd} u^d ,
\]

to the 4-velocity vector. In order to proceed, one needs to substitute (3.1.1) into the above, together with decompositions (2.2.2) and (2.2.7). Then, employing the auxiliary relations (2.3.5)-(2.3.7), the timelike and spacelike parts of the resulting expression result into three propagation formulae and three constraints respectively. More specifically, the trace of the timelike component provides the Raychaudhuri equation

\[
\dot{\Theta} = -\frac{1}{3} \Theta^2 - \frac{1}{2} (\rho + 3p) - 2(\sigma^2 - \omega^2) + D^a A_a + A_a A^a + \Lambda ,
\]

which dictates the evolution of the volume scalar. On the other hand, the symmetric traceless part leads to the propagation formula of the shear

\[
\dot{\sigma}_{(ab)} = -\frac{2}{3} \Theta \sigma_{ab} - \sigma_{c(a} \sigma^{c}b) - \omega_{(a} \omega_{b)} + D_{(a} A_{b)} + A_{(a} A_{b)} - E_{ab} + \frac{1}{2} \pi_{ab}
\]
and its antisymmetric counterpart governs the vorticity evolution

\[ \dot{\omega}_{(a)} = -\frac{2}{3} \Theta \omega_a - \frac{1}{2} \text{curl } A_a + \sigma_{ab} \omega^b. \]  

(3.1.5)

Note that \( \sigma^2 = \sigma_{ab} \sigma^{ab}/2 \) and \( \omega^2 = \omega_{ab} \omega^{ab}/2 = \omega_a \omega^a \) are the (square) magnitudes of the shear and the vorticity respectively. Also, \( E_{ab} \) is the electric Weyl tensor (see § 2.2 earlier) and \( \text{curl } v_a = \varepsilon_{abc} D^b v^c \) for every spacelike vector \( v_a \) by construction. Then, \( D^b \omega_{ab} = \text{curl } \omega_a \).

The above propagation formulae are supplemented by three constraints derived from the spacelike component of Eq. (3.1.2). In particular, the trace, the symmetric traceless and the antisymmetric parts of (3.1.2) lead to

\[ D^a \omega_a = A_a \omega^a, \]  

(3.1.6)

\[ H_{ab} = \text{curl } \sigma_{ab} + D_{(a} \omega_{b)} + 2 A_{(a} \omega_{b)} \]  

(3.1.7)

and

\[ D^b \sigma_{ab} = \frac{2}{3} D_a \Theta + \text{curl } \omega_a + 2 \varepsilon_{abc} A^b \omega^c - q_a, \]  

(3.1.8)

respectively.

Of all the kinematic formulae given above, the Raychaudhuri’s equation is undoubtedly the most widely used, with applications in both astrophysics and cosmology (see [44] and also [? ,?] for reviews). Since Eq. (3.1.3) monitors the average separation between the worldlines of neighbouring observers, it has been readily applied to gravitational-collapse studies as well as to those of the universal expansion. Negative terms on the right-hand side of (3.1.3) assist the collapse and decelerate the expansion, while positive ones resist contraction and accelerate the expansion. This explains why the Raychaudhuri equation has been central in all the singularity theorems (see [66,69] and references therein) and also involved in all the accelerated-expansion scenarios (e.g. see [21,22]).

3.2 Conservation laws

Contracting the (once contracted) Bianchi identities (see Eq. (2.2.5) in § 9.2.1) again, leads to the constraint

\[ \nabla^b G_{ab} = \nabla^b R_{ab} - \frac{1}{2} \nabla_a R = 0, \]  

(3.2.1)

Combining the above with the Einstein field equations (see (2.2.1 in § 9.2.1) guarantees energy-momentum conservation, namely that \( \nabla^b T_{ab} = 0 \). The latter splits into a timelike and a spacelike component, which respectively provide the conservation laws for the energy and the momentum densities. Written for the case of an imperfect fluid, the former reads

\[ \dot{\rho} = -\Theta (\rho + p) - D^a q_a - 2 A^a q_a - \sigma^{ab} \pi_{ab}. \]  

(3.2.2)
The momentum-density conservation law, on the other hand, satisfies the Navier-Stokes equation, that is

\[(\rho + p)A_a = -D_a p - \dot{q}_a - \frac{4}{3} \Theta q_a - (\sigma_{ab} + \omega_{ab})q^b - D^b \pi_{ab} - \pi_{ab}A^b. \quad (3.2.3)\]

Applied to a perfect fluid, with \(q_a = 0 = \pi_{ab}\), these two formulae reduce to

\[\dot{\rho} = -\Theta (\rho + p) \quad \text{and} \quad (\rho + p)A_a = -D_a p, \quad (3.2.4)\]

respectively. Note that, according to (3.2.4b), the sum \(\rho + p\) measures the relativistic (effective) inertial mass/energy of the medium.

### 3.3 Spatial curvature

When there is no rotation the instantaneous 3-dimensional rest-spaces of the fundamental observers form a single spacelike hypersurface orthogonal to their worldlines. These are the hypersurfaces of simultaneity for the aforementioned observers. In the presence of vorticity, on the other hand, the Frobenius theorem forbids the existence of such integrable 3-D hypersurfaces (e.g. see [69,70]). In such a case, the observers’ rest-spaces no longer mesh together to form their hypersurface of simultaneity.

The curvature of the 3-D spatial sections is monitored by the projected Riemann tensor, which is defined by

\[ \mathcal{R}_{abcd} = h_a^q h_b^s h_c^f h_d^p R_{qsfp} - v_{ac} v_{bd} + v_{ad} v_{bc}, \quad (3.3.1)\]

where \(v_{ab} = D_b u_a\). Then, on using expressions (2.2.1)-(2.3.5) and decompositions (2.2.2), (2.2.7), we find [62]

\[
\mathcal{R}_{abcd} = -\varepsilon_{abq} \varepsilon_{cds} E^{qs} + \frac{1}{3} \left(\rho - \frac{1}{3} \Theta^2 + \Lambda\right) (h_{ac} h_{bd} - h_{ad} h_{bc}) \\
+ \frac{1}{2} (h_{ac} \pi_{bd} + \pi_{ac} h_{bd} - h_{ad} \pi_{bc} - \pi_{ad} h_{bc}) \\
- \frac{1}{3} \Theta [h_{ac} (\sigma_{bd} + \omega_{bd}) + (\sigma_{ac} + \omega_{ac}) h_{bd} - h_{ad} (\sigma_{bc} + \omega_{bc}) - (\sigma_{ad} + \omega_{ad}) h_{bc}] \\
- (\sigma_{ac} + \omega_{ac}) (\sigma_{bd} + \omega_{bd}) + (\sigma_{ad} + \omega_{ad}) (\sigma_{bc} + \omega_{bc}), \quad (3.3.2)
\]

which decomposes the 3-Riemann tensor into its irreducible components. As mentioned above, when \(\omega_a = 0\) \(\mathcal{R}_{abcd}\) is the curvature tensor of the hypersurfaces of simultaneity (orthogonal to \(u_a\)). In analogy to their 4-dimensional counterparts, the projected 3-Ricci tensor and the corresponding 3-Ricci scalar are defined by

\[ \mathcal{R}_{ab} = h^{cd} \mathcal{R}_{acbd} = \mathcal{R}^c_{acb} \quad \text{and} \quad \mathcal{R} = h^{ab} \mathcal{R}_{ab}, \quad (3.3.3)\]

respectively.
Starting from (3.3.2), one arrive at the algebraic symmetries of $R_{abcd}$. More specifically, we find
\[ R_{abcd} = R_{[ab][cd]} \quad (3.3.4) \]
and
\[ R_{abcd} - R_{cdab} = -\frac{2}{3} \Theta \left( h_{ac} \omega_{bd} + \omega_{ac} h_{bd} - h_{ad} \omega_{bc} - \omega_{ad} h_{bc} \right) - 2 (\sigma_{ac} \omega_{bd} + \omega_{ac} \sigma_{bd} - \sigma_{ad} \omega_{bc} - \omega_{ad} \sigma_{bc}) . \quad (3.3.5) \]
The latter relation ensures that $R_{abcd} = R_{cdab}$ in the absence of vorticity, in which case the spatial Riemann tensor satisfies all the symmetries of its 4-dimensional counterpart.

Contracting (3.3.2) along the first and third indices provides an expression for the 3-Ricci tensor, namely the Gauss-Codacci formula
\[ R_{ab} = E_{ab} + \frac{2}{3} \left( \rho - \frac{1}{3} \Theta^2 + \sigma^2 - \omega^2 + \Lambda \right) h_{ab} + \frac{1}{2} \pi_{ab} - \frac{1}{3} \Theta (\sigma_{ab} + \omega_{ab}) + \sigma_{c(a} \sigma_{b)} - \omega_{c(a} \omega_{b)} + 2 \sigma_{c[a} \omega_{b]} , \quad (3.3.6) \]
while a further contraction leads to the 3-Ricci scalar and the generalised Friedmann equation
\[ R = h^{ab} R_{ab} = 2 \left( \rho - \frac{1}{3} \Theta^2 + \sigma^2 - \omega^2 + \Lambda \right) . \quad (3.3.7) \]
Finally, combining Eqs. (3.3.6) and (3.3.7) we obtain
\[ R_{ab} = \frac{1}{3} R h_{ab} + E_{ab} + \frac{1}{2} \pi_{ab} - \frac{1}{3} \Theta (\sigma_{ab} + \omega_{ab}) + \sigma_{c(a} \sigma_{b)} - \omega_{c(a} \omega_{b)} + 2 \sigma_{c[a} \omega_{b]} . \quad (3.3.8) \]
Note that, in contrast to its 4-D counterpart, the 3-Ricci tensor is not necessarily symmetric. In particular, $R_{[ab]} \neq 0$ when $\omega_{ab} \neq 0$. Also, with the exception of the first, all terms on the right-hand side of the above are traceless.

### 3.4 Weyl curvature

Splitting the once contracted Bianchi identities (see Eq. (2.2.5) in § 2.2) into their timelike and spacelike parts leads to a set of two propagation and two constraint equations. These govern the long-range gravitational field, that is tidal forces and gravity waves. More specifically, employing decomposition (2.2.7), the timelike component of (2.2.5) gives [55]
\[ \dot{E}_{(ab)} = -\Theta E_{ab} - \frac{1}{2} (\rho + p) \sigma_{ab} + \text{curl} H_{ab} - \frac{1}{2} \pi_{ab} - \frac{1}{6} \Theta \pi_{ab} - \frac{1}{2} D_{(a} q_{b)} - A_{(a} q_{b)} + 3 \sigma_{c(a}^\varepsilon \left( E_{b) c} - \frac{1}{6} \pi_{b) c} \right) + \varepsilon_{cd(a} \left[ 2 A^c H_{b)}^d - \omega^c \left( E_{b)}^d + \frac{1}{2} \pi_{b) d} \right] \right] \quad (3.4.1) \]
and

\[
\dot{H}_{ab} = -\Theta H_{ab} - \text{curl} E_{ab} + \frac{1}{2} \text{curl} \pi_{ab} + 3\sigma_{(a}^c H_{b)c} - \frac{3}{2} \omega_{(a} q_{b)} - \varepsilon_{cd(a} \left( 2A_c^e E_b^d - \frac{1}{2} \sigma_{b)}^e q^d + \omega^e H_b^d \right) .
\]  

(3.4.2)

The time derivatives of the above lead to a pair of wavelike equations monitoring the electric and the magnetic Weyl tensors and showing how gravitational waves travel like ripples in the spacetime fabric. These waves also obey a set of constraints, which follow from the spacelike component of Eq. (2.2.5) and assume the form

\[
D^b E_{ab} = \frac{1}{3} D_a \rho - \frac{1}{2} D^b \pi_{ab} - \frac{1}{3} \Theta q_a + \frac{1}{2} \sigma_{ab} q^b - 3H_{ab} \omega^b + \varepsilon_{abc} \left( \sigma^b_d H^{cd} - \frac{3}{2} \omega^b q^c \right)
\]  

(3.4.3)

and

\[
D^b H_{ab} = (\rho + p) \omega_a - \frac{1}{2} \text{curl} q_a + 3E_{ab} \omega^b - \frac{1}{2} \pi_{ab} \omega^b - \varepsilon_{abc} \sigma^b_d \left( E^{cd} + \frac{1}{2} \pi^{cd} \right),
\]  

(3.4.4)

respectively [55]. Expressions (3.4.1)-(3.4.4) share a close resemblance with Maxwell’s equations, which explains the names adopted for \(E_{ab}\) and \(H_{ab}\). The Maxwell-like form of the free gravitational field reflects a rich correspondence between electromagnetism and general relativity that has been the subject of theoretical debate for decades (e.g. see [?, ?, 54, ?]).
4 The Friedmann universes

So far, we have considered inhomogeneous and anisotropic cosmological spacetimes, containing with a general imperfect fluid. The high isotropy of the CMB, however, together with our theoretical prejudice, namely the Copernican principle, strongly support a universe that is homogeneous and isotropic on cosmological scales. Put another way, our universe has the characteristics of a Friedmann-Robertson-Walker (FRW) model.

4.1 The FRW metric

The simplest non-static, non vacuum solution of the Einstein field equations is described by the Robertson-Walker line element. In spherical (comoving) coordinates the latter reads

$$ds^2 = -dt^2 + a^2 \left[ dr^2 + f_K^2(\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (4.1.1)$$

where $a = a(t)$ is the scale factor and the function $f_K = f_K(r)$ depends on the geometry of the 3-dimensional spatial hypersurfaces. The scale cosmological factor defines a characteristic length scale and leads to the familiar Hubble parameter by means of $H = 3a/t$. The latter determines the rate of the (isotropic) expansion. The maximal symmetry of the FRW 3-space ensures that the associated 3-Ricci scalar is given $R = 6K/a^2$, where $K = 0, \pm 1$ is the 3-curvature index. In particular,

$$f_K(r) = \begin{cases} 
\sin r & \text{for } K = +1, \\
r & \text{for } K = 0, \\
\sinh r & \text{for } K = -1.
\end{cases} \quad (4.1.2)$$

When $K = +1$, the spatial sections are finite and have spherical geometry. Otherwise, the 3-D hypersurfaces are Euclidean (when $K = 0$), or hyperbolic (for $K = -1$). In either of the last two cases, the 3-space is infinite unless nontrivial topologies are employed.

4.2 FRW cosmologies

The high symmetry of the Friedmann spacetimes can accommodate only scalars that are functions of time only. Quantities that represent anisotropy or inhomogeneity are identically zero. Therefore, FRW models have $\Theta = 3H(t) \neq 0$, $\sigma_{ab} = 0 = \omega_a = A_a$, $E_{ab} = 0 = H_{ab}$, with $H = \dot{a}/a$ representing the Hubble parameter. For the same reasons, the Friedmann models can only contain matter in the perfect-fluid form (with $\rho = \rho(t)$ and $p = p(t)$). Moreover, given the homogeneity of the 3-space, all orthogonally projected gradients (e.g. $D_a\rho$, $D_a p$, etc) vanish by default. As a result, the only nontrivial equations are Raychaudhuri's formula, the
continuity equation and the Friedmann equation. These follow from (3.1.3), (3.2.2) and (3.3.7) and take the form

\[
\dot{H} = -H^2 - \frac{1}{6} (\rho + 3p) + \frac{1}{3} \Lambda, \quad \dot{\rho} = -3H(\rho + p)
\]  

(4.2.1)

and

\[
H^2 = \frac{1}{3} \rho - \frac{K}{a^2} + \frac{1}{3} \Lambda,
\]

(4.2.2)

respectively. It should also be noted that the maximal symmetry of the FRW spatial hypersurfaces implies that Eq. (3.3.2) reduces to \( R_{abcd} = (K/a^2)(h_{ac}h_{bd} - h_{ad}h_{bc}) \).

On introducing the density parameter \( \Omega_\rho = \rho/3H^2 \), together with its effective counterparts \( \Omega_\Lambda = \Lambda/3H^2 \) and \( \Omega_K = -K/(aH)^2 \), the Friedmann equation (see (4.2.2) above) reads

\[
1 = \Omega_\rho + \Omega_K + \Omega_\Lambda.
\]

(4.2.3)

Accordingly, when \( \Lambda = 0 \), the Friedmann equation reduces to \( K/a^2 = H^2(\Omega_\rho - 1) \). In that case the 3-space is flat (i.e. \( K = 0 \)) when \( \Omega_\rho = 1 \) and the density of the matter takes the critical value \( \rho = \rho_c = 3H^2 \). Alternatively, \( \Omega_\rho > 1 \) and \( \Omega_\rho < 1 \) lead to spherical and hyperbolic spatial geometry, with \( K = +1 \) and \( K = -1 \) respectively.

One may also put Eqs. (4.2.1a) and (4.2.2) together to arrive at the following alternative expression of the Raychadhuri equation

\[
\dot{H} = -\frac{1}{2} (\rho + p) + \frac{K}{a^2},
\]

(4.2.4)

with an explicit 3-curvature dependence. Finally, by employing some straightforward algebra, Raychaudhuri’s formula (see Eq. (4.2.1a) in § 4.2) recasts as

\[
qH^2 = \frac{1}{6} (\rho + 3p) - \frac{1}{3} \Lambda,
\]

(4.2.5)

where \( q = -\dot{a}/a^2 = -[1 + (\dot{H}/H^2)] \) is the dimensionless deceleration parameter. When the latter takes negative values the universal expansion is accelerated. Consequently, exact FRW models with zero cosmological constant must violate the strong energy condition (i.e. they must contain matter with \( \rho + 3p < 0 \)) if their expansion is to accelerate.

Note that the expansion rate of the universe defines an additional characteristic length scale, namely the Hubble radius (or the Hubble horizon), by means of

\[
\lambda_H = H^{-1}.
\]

(4.2.6)

In Friedmann models with conventional matter that satisfies the strong energy condition, the scale-factor evolution (see § 4.3 next) ensures that the Hubble scale essentially coincides with the particle horizon and therefore with the observable universe (i.e. \( \lambda_H \propto t \)). In that case, the Hubble radius also determines the domain of causal contact.
The scale factor of an FRW spacetime with non-Euclidean 3-geometry also determines the curvature scale \((\lambda_K = a)\) of the associated universe. This is the threshold beyond which any deviations from Euclidean flatness that may exist start becoming important (e.g. see [71]). The curvature scale and the Hubble horizon are related by Eq. (4.2.2), which in the absence of \(\Lambda\) takes the form

\[
\left(\frac{\lambda_K}{\lambda_H}\right)^2 = -\frac{K}{1 - \Omega}\,.
\]  

Hence, \(\lambda_K > \lambda_H\) always when \(K = -1\), with \(\lambda_K \to \infty\) as \(\Omega \to 1\) and \(\lambda_K \to \lambda_H\) for \(\Omega \to 0\). In closed models, on the other hand, Eq. (4.2.7) implies that \(\lambda_K > \lambda_H\) when \(\Omega < 2\) and \(\lambda_K \leq \lambda_H\) if \(\Omega \geq 2\). We should finally note that the importance of spatial curvature within a comoving section does not change, since \(\lambda_K\) simply redshifts with the universal expansion.

### 4.3 Scale-factor evolution in FRW cosmologies

In order to close the system of (4.2.1) and (4.2.2) one needs an equation of state for the matter. In what follows we will consider barotropic perfect fluids in the form of pressureless dust and isotropic radiation (with \(p = 0\) and \(p = \rho/3\) respectively). Setting \(w = p/\rho\) as the barotropic index of the cosmic medium (with \(\_\_w = 0\)), the continuity equation (see expression (4.2.1b) in § 4.2 earlier) gives

\[
\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}.
\]  

Substituting the above into the Friedmann equation, assuming Euclidean spatial sections, zero cosmological constant (i.e. setting \(K = 0 = \Lambda\)) and normalising our initial conditions so that \(a(t = 0) = 0\), we obtain

\[
a = a_0 \left(\frac{t}{t_0}\right)^{2/3(1+w)},
\]  

with \(w \neq -1\). Having normalised our solution so that \(a(t = 0) = 0\). For non-relativistic matter, in the form of baryonic ‘dust’ or non-baryonic “cold dark matter” (CDM) with \(w = 0\), we find \(a \propto t^{2/3}\) and recover the familiar Einstein-de Sitter solution. Alternatively, (4.3.2) leads to \(a \propto t^{1/2}\) in the case radiation (when \(w = 1/3\)) and \(a \propto t^{1/3}\) for “stiff matter” with \(w = 1\).

A special case is that of matter with \(w = -1/3\) and zero effective gravitational mass, which leads to the so-called “coasting universe” with \(a \propto t\).

It should be emphasised that solution (4.3.2) does not apply to media with \(w = -1\). In that case, Eq. (4.3.1) ensures that \(\rho = \rho_0 = \text{constant}\), which substituted into (4.2.2) guarantees thats \(H = H_0 = \text{constant}\) and subsequently leads to exponential (inflationary) expansion with \(a \propto e^{H_0(t-t_0)}\). We also note that throughout a phase of exponential (de Sitter) inflation, the Hubble horizon \((\lambda_H)\) remains constant, whereas at the same time the particle horizon \((\lambda_P)\) increases in the usual manner.

The continuity equation has no 3-curvature dependence, which implies that (4.3.1) monitors the evolution of the matter density in all three FRW models. When the latter have non-Euclidean spatial geometry, it helps to parametrise them terms of the conformal time \((\eta -\)
with $\dot{\eta} = 1/a$. Then, when $K = +1$ and $\Lambda = 0$ relations (4.2.1), (4.2.2) solve to give

$$a = a_0 \left\{ \frac{\sin[(1 + 3w)\eta/2 + C]}{\sin[(1 + 3w)\eta_0/2 + C]} \right\}^{2/(1+3w)}, \quad (4.3.3)$$

where $w \neq -1/3$ and $(1 + 3w)\eta/2 + C \in (0, \pi)$. Setting $a(\eta \to 0) \to 0$ initially, the value $\eta = \pi/(1 + 3w)$ corresponds to the moment of maximum expansion, with $a = a_{\text{max}} = a_0\{\sin[(1 + 3w)\eta_0/2]\}^{2/(1+3w)}$. It is straightforward to show that solution (4.3.3) reduces to $a \propto \sin^2(\eta/2)$ [72] for pressure-free matter, while for radiation we obtain $a \propto \sin \eta$. Also note that (4.3.3) no longer applies when $w = -1/3$. In such a case, Eq. (4.2.1a) leads immediately to the familiar coasting solution with $a \propto t$. Finally, expressions (4.2.1b) and (4.2.2) relate the scale factor to the proper time when $w = -1$. As in spatially flat models, Eq. (4.2.1b) guarantees that $\rho = \rho_0 = \text{constant}$ and then, by means of (4.2.2), it leads to $a(1 + \sqrt{3/\rho_0 H}) \propto e^{\sqrt{\rho_0/3} t}$.

On introducing the conformal time to FRW cosmologies with hyperbolic spatial geometry, zero cosmological constant and $w \neq -1/3$, we arrive at

$$a = a_0 \left\{ \frac{\sinh[(1 + 3w)\eta/2 + C]}{\sinh[(1 + 3w)\eta_0/2 + C]} \right\}^{2/(1+3w)}, \quad (4.3.4)$$

with $(1 + 3w)\eta/2 + C > 0$ in this case. Not surprisingly, the above solution can be also obtained from (4.3.3), by replacing the trigonometric functions with their hyperbolic counterparts. Then, for universes dominated by pressureless ‘dust’ and by relativistic species, we have $a \propto \sinh^2(\eta/2)$ and $a \propto \sinh \eta$ respectively [72]. Finally, similarly to the $K = +1$ case, Eqs. (4.2.1) and (4.2.2) guarantee that $a \propto t$ when $w = -1/3$ and $a(1 + \sqrt{3/\rho_0 H}) \propto e^{\sqrt{\rho_0/3} t}$ for $w = -1$. 

24
5 Inhomogeneous cosmologies

The Friedmann models have uniform spatial sections, namely their 3-D space is homogeneous and isotropic. The universe we live in, on the other hand, is full of structure. Consequently, to reproduce the observable cosmos, we need models with considerably less symmetry, which means that we need to employ inhomogeneous and anisotropic cosmologies.

5.1 Gauge-invariant inhomogeneities

Covariantly, inhomogeneities in the spatial distribution of physical quantities are described by their projected gradients. When studying structure formation the key variable gradient of the matter energy density, which is defined by

\[ \Delta_a = \frac{a}{\rho} D_a \rho. \] (5.1.1)

This dimensionless 3-vector monitors density variations between neighbouring observers [73]. Moreover, \( \Delta_a \) is identically zero in the homogeneous spatial sections of the FRW backgrounds (since \( D_a \rho \) vanishes there), which makes it gauge invariant at the linear perturbative level, in line with the Stewart and Walker lemma [74].

The density gradient is typically supplemented by an auxiliary variable describing spatial inhomogeneities in the volume-expansion gradients. Following [73], the latter is given by

\[ Z_a = a D_a \Theta \] (5.1.2)

and complies with the Stewart and Walker criterion for linear gauge invariance (since \( D_a \Theta = 0 \) in a Friedmann background by default).

5.1.1 Three types of inhomogeneity

The density gradient \( \Delta_a \) carries collective information on three types of inhomogeneities, namely a scalar, vector and tensor-like perturbations. The former monitor overdensities or underdensities in the matter distribution, the vector perturbations describe rotational (vortex-like) distortions and the tensor modes are related to shape-changes in the density distribution (e.g. from spherical to ellipsoidal). All the information is encoded in the projected gradient \( \Delta_{ab} = a D_b \Delta_a \) and it is decoded by splitting the second-rank tensor \( \Delta_{ab} \) into its irreducible parts as follows [75]

\[ \Delta_{ab} = \frac{1}{3} \Delta h_{ab} + \Delta_{(ab)} + \Delta_{[ab]} . \] (5.1.3)

In the above, the scalar \( \Delta = a D^a \Delta_a \) monitors overdensities/underdensities, the symmetric traceless tensor \( \Delta_{(ab)} = a D_{(b} \Delta_{a)} \) describes anisotropic patterns in the matter distribution,
such as of pancakes or cigar-like structures, while the antisymmetric tensor \( \Delta_{[ab]} = aD_b \Delta_a \) (which is a vector for all practical purposes) is related to vortex-like distortions.

It goes without saying that an exactly analogous decomposition applies to the projected gradient \( Z_{ab} = aD_b Z_a \) of the expansion inhomogeneities. More specifically, we have

\[
Z_{ab} = \frac{1}{3} Z h_{ab} + Z_{(ab)} + Z_{[ab]},
\]

(5.1.4)

with \( Z = aD^a Z_a \), \( Z_{(ab)} = aD_{(b} Z_{a)} \) and \( Z_{[ab]} = aD_{[b} Z_{a]} \) by definition.

5.2 Inhomogeneous single-fluid cosmologies

The observed large-scale structure of the universe is believed to result from Jeans-type instabilities, when initially small inhomogeneities in the density distribution of the cosmic medium grow gravitationally to form the galaxies and the voids we see today. Here, we will provide the key nonlinear equations governing structure formation in single-fluid cosmologies.

5.2.1 Imperfect fluids

Let us assume a general spacetime containing a single imperfect fluid. Taking the spatial covariant derivative of \( \Delta_a \), while employing the energy and momentum conservation laws (see Eqs. (3.2.2) and (3.2.3) earlier), gives

\[
\dot{\Delta}_a = \frac{\rho}{\rho} \Theta \Delta_a - \left(1 + \frac{\rho}{\rho}\right) Z_a + \frac{\alpha \Theta}{\rho} \left(q_a + \frac{4}{3} \Theta q_a\right) - \frac{a}{\rho} D_a D^b q_b + \frac{a \Theta}{\rho} D^b \pi_{ab} \\
- \left(\sigma^b + \omega^b_a\right) \Delta_b - \frac{a}{\rho} D_a \left(2A^b q_b + \sigma^{bc} \pi_{bc}\right) + \frac{a \Theta}{\rho} \left(\sigma_{ab} + \omega_{ab}\right) q^b + \frac{a \Theta}{\rho} \pi_{ab} A^b \\
+ \frac{1}{\rho} \left(D^b q_b + 2A^b q_b + \sigma^{bc} \pi_{bc}\right) \left(\Delta_a - a A_a\right),
\]

(5.2.1)

Similarly, the time derivative of \( Z_a \), together with the Raychaudhuri equation, lead to the following propagation formula of the expansion gradients

\[
\dot{Z}_a = -\frac{2}{3} \Theta Z_a - \frac{1}{2} \rho \Delta_a - \frac{3}{2} aD_a p - a \left[\frac{1}{3} \Theta^2 + \frac{1}{2} (\rho + 3p) \right] A_a + aD_a D^b A_b \\
- \left(\sigma^b + \omega^b_a\right) \Delta_b - 2a D_a \left(\sigma^2 - \omega^2\right) + 2a A^b D_a A_b \\
- a \left[2 \left(\sigma^2 - \omega^2\right) - D^b A_b - A^b A_b\right] A_a.
\]

(5.2.2)
5.2.2 Perfect fluids

In the case of a perfect fluid, the energy-flux and the anisotropic pressure vanish identically (i.e. $q_a = 0 = \pi_{ab}$). This considerably simplifies Eq. (5.2.1), which reduces to

$$\dot{\Delta}_a = \frac{p}{\rho} \Theta \Delta_a - \left(1 + \frac{p}{\rho}\right) \mathcal{Z}_a - \left(\sigma^b_a + \omega^b_a\right) \Delta_b. \quad (5.2.3)$$

On the other hand, the propagation equation of the expansion gradients, which retains the form of (5.2.2). However, the 4-acceleration is now given by (3.2.4b) instead of (3.2.3). If, in addition, the medium is barotropic (with $p = p(\rho)$), the pressure and the density gradients are directly related by $D_a \rho = c_s^2 D_a \rho$, where $c_s^2 = \rho/\rho$ represents the adiabatic sound speed.

5.3 Linear perturbations

During the early stages of structure formation, perturbations are weak. For instance, the magnitude of $\Delta_a$ is much smaller than unity (i.e. $\sqrt{\Delta_a \Delta^a} \ll 1$) throughout the linear regime. As long as the perturbations remain weak, the nonlinear equations simplify considerably.

5.3.1 The linearisation scheme

The process of linearisation starts once the unperturbed background cosmology has been decided, which in our case will always be a homogeneous and anisotropic Friedmann cosmology. Then, variables with nonzero background value will be assigned zero perturbative order, while those that vanish there will be first-order perturbations [73,76]. This also guarantees that all linear variables satisfy the Stewart-Walker criterion for gauge-invariance [74].

By construction, all the first-order variables are weak relative to the background quantities and have perturbative order $O(\epsilon)$, where $\epsilon$ is the “smallness parameter” (e.g. see [77]). When linearising the full equations, all terms that are products of $O(\epsilon)$ variables have higher than the first perturbative order and are therefore neglected.

When the background model is identified with the homogeneous and isotropic FRW spacetimes, the only zero-order quantities are the energy density of the matter ($\rho$), its isotropic pressure ($p$) and the volume expansion scalar ($\Theta = 3H$ – recall that $H = \dot{a}/a$ is the cosmological scale factor). When the background spatial sections have non-Euclidean geometry, the 3-Ricci scalar ($R$) is also of zero perturbative order. The rest of the variables (e.g. $A_a$, $\Delta_a$, $\sigma_{ab}$, $E_{ab}$, etc) are first order perturbations due to the spatial homogeneity or/and the isotropy of the Friedmann universes. Recall that the high symmetry of the FRW spacetimes ensures that only time-dependent scalars survive there.
5.3.2 The linear relations

Assuming a perturbed, almost-FRW background filled with an imperfect fluid, its kinematic evolution (see § 3.1 previously) is monitored by the first-order expressions

\[ \dot{\Theta} = -\frac{1}{3} \Theta^2 - \frac{1}{2} \rho (1 + 3w) + D^a A_a + \Lambda, \quad (5.3.1) \]

\[ \dot{\sigma}_{ab} = -2H\sigma_{ab} + D_{(a} A_{b)} - E_{ab} + \frac{1}{2} \pi_{ab} \quad (5.3.2) \]

and

\[ \dot{\omega}_a = -2H\omega_a - \frac{1}{2} \text{curl} A_a, \quad (5.3.3) \]

to linear order, where \( \sigma_{ab} \) and \( \omega_a \) are defined in § 3.1.

Since \( \dot{\sigma}_{(ab)} = \dot{\sigma}_{ab} \) and \( \dot{\omega}_{(a)} = \dot{\omega}_a \) at the linear perturbative level. We also remind the reader that \( w = p/\rho \) is the barotropic index of the cosmic medium. At the same time, the associated constraints (see relations (3.1.6)-(3.1.8) in § 3.1) respectively reduce to

\[ D^a \omega_a = 0, \quad (5.3.4) \]

\[ H_{ab} = \text{curl} \sigma_{ab} + D_{(a} \omega_{b)} \quad (5.3.5) \]

and

\[ D^b \sigma_{ab} = \frac{2}{3} D_a \Theta + \text{curl} \omega_a - q_a. \quad (5.3.6) \]

Similarly, the energy-density and the momentum-density conservation laws (see Eqs. (3.2.2) and (3.2.3) in § 3.2) linearise to

\[ \dot{\rho} = -3H\rho (1 + w) - D^a q_a \quad (5.3.7) \]

and

\[ \rho (1 + w) A_a = -D_a p - \dot{q}_a - 4H q_a - D_b \pi_{ab}, \quad (5.3.8) \]

respectively (since \( \dot{q}_{(a)} = \dot{q}_a \) to first approximation).

Analogous first-order relations also follow from all the nonlinear expressions given in § 3 and § 5.2 earlier. Here, we only need to provide the linear versions of (5.2.1) and (5.2.2) derived in § 5.2.1. These read

\[ \dot{\Delta}_a = 3wH \Delta_a - (1 + w) \Delta_a + \frac{3aH}{\rho} (q_a + 4H q_a) - \frac{a}{\rho} D_a D^b q_b + \frac{3aH}{\rho} D^b \pi_{ab} \quad (5.3.9) \]

and

\[ \dot{\Delta}_{(a)} = \dot{\Delta}_a \quad (5.3.10) \]

to linear order.
5.4 Linear solutions

When the material content of the universe has the form of a perfect fluid (i.e. for $q_a = 0 = \pi_{ab}$), the system of (5.3.9) and (5.3.10) solves analytically, both during the dust and the radiation epochs of the universe. Here, we provide the dust-era solution for completeness and illustration purposes, referring the reader to [21,22] for further information and discussion.

Going back to Eqs. (5.3.9) and (5.3.10), setting $p = 0$, $w = p/\rho = 0$ and $A_a = 0$ (see expression (5.3.8) before), we arrive at the simplified system

$$
\dot{\Delta}_a = -\dot{Z}_a \quad \text{and} \quad \dot{Z}_a = -2H Z_a - \frac{1}{2} \rho \Delta_a .
$$

(5.4.1)

Recalling that $H = 2/3t$ after matter-radiation equality, which means that $H = 2/3t$ and $\rho = 4/3t^2$, the above set solves to give

$$
\Delta_a \propto t^{2/3} \propto a \quad \text{and} \quad Z_a \propto t^{-1/3} \propto a^{-1/2} ,
$$

(5.4.2)

on all scales. Consequently, after equipartition, linear inhomogeneities in the density distribution of the (pressureless) matter grow as $\Delta_a \propto a$, that is in tune with the increasing dimensions of the universe.
6 “Tilted” cosmologies

No real observer in the universe follows the smooth universal expansion, commonly referred to as the “Hubble flow”. Instead, we all have peculiar velocities relative to it. Our Milky Way and the Local Group of galaxies, for example, drift at a speed in excess of 600 km/sec.

6.1 Idealised and real observers

Large-scale peculiar motions, also known as “bulk flows” are commonplace in our universe. In addition to the aforementioned motion of our Local Group, there have been continuous reports of large-scale peculiar motions with typical sizes of few hundred Mpc and speeds of few hundred km/sec [7–9,12]. Despite their verified presence, however, most of the available theoretical cosmological studies bypass them. Moreover, those few studies that include peculiar motions are usually Newtonian in nature and typically take the viewpoint of the idealised (Hubble-flow) observers. As a result, the role and the implications of these bulk peculiar flows for the large-scale structure and the large-scale kinematics of the universe remains largely unaccounted for.

Theoretical studies of relative motions require two (at least) coordinate systems, moving relative to each other with finite peculiar velocity. One of the frames will act as the reference coordinate system, with respect to which peculiar velocities can be defined and measured. The other will be the rest-frame of the relatively moving observers. In cosmology, the former has been typically identified with the idealised coordinate system of the (fictitious) Hubble-flow observers, that is with the CMB frame, while the latter is associated with the rest-frame of the real observers that live in typical galaxies like the Milky Way. This is also the frame where observations take place.

In contrast to Newtonian physics, Einstein’s theory no longer treats space and time as absolute and separate entities, but as interconnected and frame-dependent. This means that relatively moving observers experience their own version of “reality”. More specifically, the observers measure their own space and time and generally their temporal and spatial measurements disagree.

On these grounds, let us introduce two families of observers, with 4-velocities $u_a$ and $\tilde{u}_a$. Suppose also that the latter group moves relative to the former with peculiar velocity $\tilde{v}_a$. Then, the three velocity fields are related by the familiar Lorentz transformation, so that

$$\tilde{u}_a = \tilde{\gamma}(u_a + \tilde{v}_a),$$

(6.1.1)

where $\tilde{\gamma} = (1 - \tilde{v}^2)^{-1/2}$ is the Lorentz-boost factor (with $\tilde{v}^2 = \tilde{v}_a\tilde{v}^a$ and $u_a\tilde{v}^a = 0$). Also note that both $u_a$ and $\tilde{u}_a$ are timelike, since $u_a u^a = -1 = \tilde{u}_a\tilde{u}^a$ by construction [23]. In addition, following (6.1.1) we conclude that $\tilde{u}_a u^a = -\tilde{\gamma}$. We may therefore use the inner-product of the two 4-velocity fields to define the hyperbolic “tilt” angle ($\beta$ – with $\cosh \beta = -\tilde{u}_a u^a = \tilde{\gamma} > 1$)

30
between the two frames (see Fig. 1). This explains the use of the term “tilted” to describe cosmological models equipped with more than one families of observers [23].

Alternatively, one may consider the relative motion of the reference frame \((u_a)\) with respect to the \(\tilde{u}_a\)-field. In that case the Lorentz transformation reads

\[
u_a = \gamma (\tilde{u}_a + v_a),
\]

with \(\gamma = (1 - v^2)^{-1/2}, v^2 = v_a v^a\) and \(\tilde{u}_a v^a = 0\). Note that the hyperbolic tilt angle between the two frames remains unchanged, since \(\cosh \beta = u_a \tilde{u}^a\). The latter ensures that \(\gamma = -u_a \tilde{u}_a = -\tilde{u}_a u^a = \tilde{\gamma}\) and subsequently that \(v^2 = \tilde{v}^2\). Finally, combining (6.1.1) and (6.1.2) we find that the two peculiar velocity fields are related by

\[
\tilde{v}_a = -\frac{1}{\gamma} v_a - v^2 u_a \quad \text{and} \quad v_a = -\frac{1}{\gamma} \tilde{v}_a - v^2 \tilde{u}_a.
\]

Clearly, when dealing with non-relativistic peculiar velocities, namely those with \(\tilde{v}^2 = v^2 \ll 1\) and \(\tilde{\gamma} = \gamma \simeq 1\), the relations (6.1.1)-(6.1.3) linearise to

\[
\tilde{u}_a = u_a + \tilde{v}_a \quad u_a \simeq \tilde{u}_a + v_a \quad \text{and} \quad \tilde{v}_a \simeq -v_a,
\]

respectively (as expected). Note that, although the first two of the above appear identical to the Newtonian Galilean transformation, they differ since the 4-velocity vectors \(u_a\) and \(\tilde{u}_a\) remain timelike. Finally, following (6.1.3a), we find \(h_{ab} \tilde{v}_b = \tilde{v}_a\) and \(\tilde{h}_{ab} \tilde{v}_b \neq \tilde{v}_a\), which imply that \(\tilde{v}_a\) lies on the spatial hyperspace \(S\) of the \(u_a\)-field. In an exactly analogous way, Eq. (6.1.3b) ensures that \(v_a\) lies on \(\tilde{S}\).

### 6.2 Tilted 1+3 spacetime splitting

In line with § 2.1, each of the two 4-velocity fields \(u_a\) and \(\tilde{u}_a\) introduces a unique 1+3 splitting of the host spacetime into a temporal direction parallel to it and an orthogonal spatial 3-D hypersurface (see Fig. 1). Then, the symmetric tensors

\[
h_{ab} = g_{ab} + u_a u_b \quad \text{and} \quad \tilde{h}_{ab} = g_{ab} + \tilde{u}_a \tilde{u}_b,
\]

project into the 3-D hyperspaces \(S\) and \(\tilde{S}\) respectively, since \(h_{ab} u^b = 0 = \tilde{h}_{ab} \tilde{u}^b\) by construction. Employing both 4-velocity fields and their associated projection tensors, one can then define the temporal and spatial derivative operators corresponding to the two coordinate systems. More specifically, following § 2.1, the operators

\[
dot = u^a \nabla_a \quad \text{and} \quad \dot{'} = \tilde{u}^a \nabla_a
\]

define time-derivatives in the two frames, while

\[
D_a = h_a^b \nabla_b \quad \text{and} \quad \tilde{D}_a = \tilde{h}_a^b \nabla_b,
\]

31
Fig. 1. Observer (O) with 4-velocity \( \tilde{u}_a \) and peculiar velocity \( \tilde{v}_a \), relative to the \( u_a \)-frame (see Eq. (6.1.1)). The 4-velocity fields \( u_a \) and \( \tilde{u}_a \) form the hyperbolic (tilt) angle \( \beta \) between them and also define the temporal directions of their associated observers. The 3-D hypersurfaces \( S \) and \( \tilde{S} \), on the other hand, define the rest-spaces of the aforementioned observers. In what follows, the \( u_a \)-field will be identified with the idealised coordinate system of the smooth Hubble flow, while \( \tilde{u}_a \) will set the rest-frame of the real observers living inside typical galaxies like our Milky Way.

are the corresponding spatial gradients.

The two coordinate systems defined in § 6.1 and also here are obviously equivalent and one is free to choose any of them to describe the evolution and the effects of peculiar motions. Nevertheless, in what follows, we will take the viewpoint of the tilted observer moving along with the \( \tilde{u}_a \)-field, unless stated otherwise. Our choice is motivated by the fact that all real observers in the universe are believed to move relative to the rest-frame of the smooth universal expansion. This in turn makes \( \tilde{u}_a \) the coordinate system where observations also take place and therefore the natural frame to use.

6.3 The peculiar kinematics

The kinematic evolution of the two 4-velocity fields defined in the previous section is determined by that of their irreducible components (see § 3.1 earlier). The gradient of the \( u_a \)-field, in particular, obeys decomposition (3.1.1), whereas its tilted counterpart splits as

\[
\nabla_b \tilde{u}_a = \tilde{\sigma}_{ab} + \tilde{\omega}_{ab} + \frac{1}{3} \tilde{\Theta} h_{ab} - \tilde{A}_a \tilde{u}_b ,
\]

(6.3.1)

with \( \tilde{\sigma}_{ab}, \tilde{\omega}_{ab}, \tilde{\Theta} \) and \( \tilde{A}_a \) defined as their non-tilded analogues and satisfying exactly analogous constraints (e.g. \( \tilde{\sigma}_{ab} \tilde{u}^b = 0 = \tilde{\omega}_{ab} \tilde{u}^b = \tilde{A}_a \tilde{u}^a \), etc – see § 3.1). The above monitors the total kinematics of observers living inside a bulk-flow domain (\( D \) – see Fig. 2) and moving relative to the Hubble expansion with peculiar velocity \( \tilde{v}_a \).

\footnote{Hereafter, “tildas” will always indicate variables and operators that are defined and used in the tilted frame of the relatively moving observers.}
Fig. 2. Observers \((O_1, O_2)\) living inside the bulk-flow domain \((D)\), moving with (local) peculiar velocity \(\tilde{v}_a\) relative to the Hubble expansion. When the peculiar velocity field has positive spatial divergence, \(\tilde{\vartheta} > 0\) and the bulk flow expands locally. Otherwise, we have (local) contraction.

The local bulk-flow kinematics are determined by the peculiar velocity field, the spatial gradient of which satisfies a decomposition analogous to that seen in Eqs. (3.1.1) and (6.3.1). Relative to the \(\tilde{u}_a\)-frame, for example, we have \([78,79,30]\)

\[
\tilde{D}_b \tilde{v}_a = \tilde{\zeta}_{ab} + \tilde{\varpi}_{ab} + \frac{1}{3} \tilde{\vartheta} \tilde{h}_{ab}. \tag{6.3.2}
\]

Here, \(\tilde{\zeta}_{ab} = \tilde{D}_{(b} \tilde{v}_{a)}\) is the shear tensor, \(\tilde{\varpi}_{ab} = \tilde{D}_{[b} \tilde{v}_{a]}\) is the vorticity tensor and \(\tilde{\vartheta} = \tilde{D}^a \tilde{v}_a\) is the volume expansion/contraction scalar of the peculiar flow. These monitor the local kinematics of the relative motion, as seen by observers moving along with it. For instance, the scalar \(\tilde{\vartheta}\) describes the local expansion/contraction of the bulk peculiar flow. More specifically, when \(\tilde{\vartheta}\) takes positive values, the flux-lines tangent to the peculiar velocity field diverge and the bulk flow expands locally. In the opposite case the flow lines converge, which implies local contraction (see Fig. 2).

In close analogy with shear proper, its peculiar counterpart \((\tilde{\zeta}_{ab})\) monitors changes in the shape of the bulk-flow domain. A nonzero peculiar vorticity \((\tilde{\varpi}_{ab})\), on the other hand, implies local rotation \((\text{with } \tilde{\varpi}_a = \tilde{\varpi}_{abc} \tilde{\varpi}^{bc}/2 \text{ defining the associated rotational axis})\).

One may choose the Hubble frame, instead of the tilted one, to describe the local peculiar kinematics. In such a case, decomposition (6.3.2) is replace by \([78,79]\)

\[
D_b \tilde{v}_a = \zeta_{ab} + \varpi_{ab} + \frac{1}{3} \vartheta h_{ab}, \tag{6.3.3}
\]

with \(\zeta_{ab} = D_{(b} \tilde{v}_{a)}\), \(\varpi_{ab} = D_{[b} \tilde{v}_{a]}\) and \(\vartheta = D^a \tilde{v}_a\). These are the irreducible kinematic variables of the peculiar flow, measured in the coordinate system of the smooth universal expansion. Starting from the associated definitions, one may derive the relations between the above and their tilded counterparts. The peculiar volume scalars, for example, are related by \([79]\)

\[
\tilde{\vartheta} = \vartheta - \frac{1}{3} \tilde{\gamma}^2 \tilde{v}^2 (\Theta - \tilde{\vartheta}) - \tilde{\gamma}^2 \left( \tilde{v}^2 A_a - \dot{\tilde{v}}_a \right) - \tilde{\gamma}^2 (\sigma_{ab} - \zeta_{ab}) \tilde{v}^a \tilde{v}^b. \tag{6.3.4}
\]
Applied to non-relativistic drift motions, with $\tilde{v}^2 \ll 1$ and $\tilde{\gamma} \approx 1$, the above reduces to linear relation $\tilde{\theta} \approx \bar{\theta}$ between the two local volume scalars.

6.4 The peculiar Raychaudhuri equation

As we mentioned above, the volume scalar $\tilde{\theta}$ (as well as its Hubble-flow counterpart $\bar{\theta}$) measures the mean bulk-flow kinematics, namely its local expansion/contraction. The local acceleration/deceleration of the bulk-flow expansion/contraction is monitored by the associated "peculiar" Raychaudhuri equation [79]. In analogy with its standard counterpart, the peculiar version of Raychaudhuri’s formula also follows from the Ricci identities. Applied to the peculiar velocity field, the latter reads

$$2\nabla_{(a} \nabla_{b)} \tilde{v}_c = R_{abcd} \tilde{u}^d,$$  \hspace{1cm} (6.4.1)

where $R_{abcd}$ is the Riemann curvature tensor (see § 2.2 for details). Contracting the above along $\tilde{u}_a$, taking the trace of the resulting expression and then using decompositions (6.3.1) and (6.3.2) provides the peculiar Raychaudhuri equation relative to the tilted frame of the bulk-flow observers, namely [79]

$$\tilde{\dot{\theta}} = -\frac{1}{3} \tilde{\Theta} \tilde{\dot{\theta}} - \tilde{\sigma}_{ab} \tilde{\omega}^{ab} + \tilde{\omega}_{ab} \tilde{\omega}^{ab} + \tilde{D}^a \tilde{v}_a' + \tilde{A}^a \tilde{v}_a' - \frac{1}{3} \tilde{\Theta} \tilde{A}^a \tilde{v}_a - \tilde{A}^a \tilde{v}^b (\tilde{\sigma}_{ab} - \tilde{\omega}_{ab}) + \tilde{A}^a \tilde{D}_a (\tilde{\gamma} \tilde{v}^2)$$

$$- R_{ab} \tilde{u}^a \tilde{v}^b.$$

(6.4.2)

The above relation determines the deceleration/acceleration of the bulk flow’s local expansion/contraction. Clearly, positive terms on the right-hand side of (6.4.2) tend to accelerate the local expansion, or decelerate the contraction. Negative terms, on the other hand, have the opposite effect.

Comparing the peculiar Raychaudhuri equation to its standard counterpart (see Eq. (3.1.3) in § 3.1), one notices analogies as well as differences. Of particular interest is the last term on the right-hand side of the above, which conveys the effects of spacetime curvature (i.e. of the gravitational field). In contrast to the curvature term seen in (3.1.3), here the Ricci tensor is not contracted twice along the observer’s temporal direction. Instead, there is contraction along the peculiar velocity field as well. This ensures that

$$R_{ab} \tilde{u}^a \tilde{v}^b = -\tilde{q}_a \tilde{v}^a - \tilde{\gamma} \tilde{v}^2 \left[ \frac{1}{2} (\tilde{\rho} + 3\tilde{\rho}) - \Lambda \right],$$

(6.4.3)

since $\tilde{v}_a = \tilde{h}_{ab} \tilde{v}_b - \tilde{\gamma} \tilde{v}^2 \tilde{u}_a$ (recall that $\tilde{u}_a \tilde{v}^a \neq 0$). Also, going back to § 2.3, we may write $R_{ab} \tilde{u}^a \tilde{v}^b = (\tilde{\rho} + 3\tilde{\rho})/2 - \Lambda$ and $\tilde{h}_{ab} \tilde{R}_{bc} \tilde{v}^c = -\tilde{q}_a$ (see expression (2.3.7a) and (2.3.7b) there). On using these auxiliary results, expression (6.4.2) recasts into [79]
\[ \ddot{v}' = -\frac{1}{3} \dot{\Theta} \dot{v}' - \tilde{\sigma}_{ab} \dot{v}' \dot{w}^{ab} + \ddot{w}_{ab} \omega^{ab} + \dddot{D}^a \dot{v}_a + \dddot{A}^a \dot{v}_a - \frac{1}{3} \dddot{\Theta} \dddot{A}^a \dot{v}_a - \dddot{\dddot{A}}^a \dot{v}^b (\tilde{\sigma}_{ab} - \dddot{\omega}_{ab}) + \dddot{A}^a \dddot{D}_a (\dot{\gamma} \dot{v}^2) \\
+ \dddot{q}_a \dddot{v}^a + \dddot{\gamma} \dddot{v}^2 \left[ \frac{1}{2} (\dddot{\rho} + 3 \dddot{p}) - \Lambda \right]. \]  

(6.4.4)

In contrast to the familiar Raychaudhuri equation (see (3.1.3) in § 3.1), the energy flux \( \dddot{q}_a \) also contributes to the peculiar version of Raychaudhuri’s formula. What is most interesting, however, is that the standard roles of the matter and of the cosmological constant (in the last term of the above) have been reversed. Indeed, as seen from the bulk-flow frame, conventional matter (with \( \dddot{p} + 3 \dddot{\rho} > 0 \)) tends to accelerate the local expansion, whereas a positive cosmological constant leads to contraction [79]. This quite intriguing and counterintuitive relative-motion effect is typically subdominant and becomes important only when the peculiar velocities are highly relativistic (i.e. for \( \dddot{v}^2 \simeq 1 \)).

Contracting (6.4.1) along the 4-velocity vector \( (u_a) \) of the Hubble-flow observers and then proceeding as before, though this time using decompositions (3.1.1) and (6.3.3), one arrives at the peculiar Raychaudhuri equation in the coordinate system of the mean universal expansion [79]

\[ \dot{\hat{\rho}} = -\frac{1}{3} \dot{\Theta} \dot{\rho} - \sigma_{ab} \dot{w}^{ab} + \omega_{ab} \omega^{ab} + \dot{D}^a \dot{v}_a + \dot{A}^a \dot{v}_a - \frac{1}{3} \dot{\Theta} \dot{A}^a \dot{v}_a - \dot{A}^a \dot{v}^b (\sigma_{ab} - \omega_{ab}) + \dddot{q}_a \dddot{v}^a. \]  

(6.4.5)

Unlike Eq. (6.4.4), here there is no direct input from the density and the pressure of the matter, or from the cosmological constant. Instead, only the energy flux contributes to the right-hand side of the above. In other words, the relative-motion effects of the local gravitational field depend on the frame choice.

Overall, the contributions of the matter density and of the pressure to the peculiar Raychaudhuri equation are either weak (for non-relativistic peculiar velocities – see expression (6.4.4)), or they vanishes altogether (see Eq. (6.4.5)). This consists a major difference, compared to the standard version of Raychaudhuri’s formula (see (3.1.3) in § 3.1), where the density and the pressure of the material component essentially dictate the mean kinematic evolution.
7 Tilted almost-FRW universes

The analysis of the previous section was fully nonlinear. In what follows we will confine our study to perturbed almost-FRW universe with a weak 4-velocity tilt. Given that in the background Friedmann model peculiar velocities vanish by default, they satisfy the Stewart & Walker criteria for linear gauge-invariance [74]. This frees our study of any gauge-related ambiguities.

7.1 Linear relations between the two frames

The irreducible kinematic variables of the three velocity fields defined in the two previous sections are related with each other. On our adopted background, the associated expressions linearise to (see [68] and also Appendix A.1 here for the list of the nonlinear relations)

\[ \tilde{\Theta} = \Theta + \tilde{v}, \quad \tilde{\sigma}_{ab} = \sigma_{ab} + \tilde{\zeta}_{ab}, \quad \tilde{\omega}_{ab} = \omega_{ab} + \tilde{\omega}_{ab} \quad \text{and} \quad \tilde{A}_a = A_a + \tilde{v}'_a + H \tilde{v}_a. \quad (7.1.1) \]

Accordingly, in a locally expanding bulk (with \( \tilde{v} > 0 \) and \( \tilde{v}/\Theta \ll 1 \)) we have \( \tilde{\Theta} \gtrsim \Theta \). Therefore, observers living inside the aforementioned bulk flow expand slightly faster than their Hubble-flow counterparts. On the other hand, if the peculiar motion is slightly contracting the local expansion rate drops (i.e. \( \tilde{\Theta} \lesssim \Theta \)).

Looking at relations (7.1.1b) and (7.1.1c) we also realise that, even if the Hubble flow is shear-free and irrotational, there will be nonzero linear shear and vorticity in the tilted frame due to relative-motion effects alone. For the same reason we will generally have \( \tilde{A}_a \neq 0 \) at the linear level, even when \( A_a = 0 \) (see Eq. (7.1.1d) above). In other words, the timelike worldlines of the tilted observers will be non-geodesics simply as a result of their peculiar motion. One might therefore say that the relative motion effects can mimic those typically attributed to non-gravitational forces.

Analogous linear expressions relate the dynamic variables measured in the Hubble and the tilted frames. Following [68] and Appendix A.1 here, we write

\[ \tilde{\rho} = \rho, \quad \tilde{p} = p, \quad \tilde{q}_a = q_a - (\rho + p)\tilde{v}_a \quad \text{and} \quad \tilde{\pi}_{ab} = \pi_{ab}, \quad (7.1.2) \]

having assumed non-relativistic peculiar velocities and an FRW background universe. The above imply that, while the Hubble-flow observers may see the cosmic medium as a perfect fluid (with \( q_a = 0 = \pi_{ab} \)), their real counterparts will see it as imperfect (with \( \tilde{q}_a = -(\rho + p)\tilde{v}_a \neq 0 \), unless \( \rho + p = 0 \)), entirely due to their relative motion. Clearly this is not the case for the density and the pressure of the matter, which remain the same in both frames.

Employing the above, one can proceed to derive the linear relations between the rest of the kinematical, dynamical and geometrical variables measured in the two frames (see [68], as well as Appendix A.1 here, for a comprehensive list). Next, we will focus our attention to the mean bulk-flow kinematics and look into the evolution of the local volume scalar (\( \tilde{\vartheta} \)). In so doing,
we will assume that the cosmic medium is a pressureless perfect fluid, with \( p = 0 = q_a = \pi_{ab} \) and set \( A_a = 0 \) due to the absence of pressure. In addition, we will take the viewpoint of the real observers that move along with the bulk peculiar flow. As far as the latter are concerned, the key linear relations are

\[
\tilde{\Theta} = \Theta + \tilde{\vartheta}, \quad \tilde{A}_a = \tilde{v}_a' + H\tilde{v}_a \quad \text{and} \quad \tilde{q}_a = -\varrho\tilde{v}_a. \quad (7.1.3)
\]

### 7.2 The linear peculiar Raychaudhuri equation

Given the spatial homogeneity and isotropy of the Friedmann models, most of the terms seen on the right-hand side of Eqs. (6.4.4) and (6.4.5) are of higher than the first perturbative order. As a result, the linear versions of these two formulae simplify considerably. Expression (6.4.4), in particular, linearises to

\[
\tilde{\vartheta}' = H\tilde{\vartheta} + \tilde{D}^a\tilde{v}_a'. \quad (7.2.1)
\]

Not surprisingly, the background expansion acts as an effective friction by slowing down the local expansion/contraction rates. The role of the source term on the right-hand side of (7.2.1), however, requires further study.

All current surveys of large-scale peculiar velocity fields (e.g. see [7-9,12]) measure the mean velocities of the associated bulk flows, but not their temporal variations. Spatial variations are also difficult to extract observationally due to the noisy velocity field. Therefore, in order to use Eq. (7.2.1), additional theoretical analysis is necessary. Turning to (relativistic) linear cosmological perturbation theory, we linearise Eq. (5.2.1) – see § 5.2.1 – in the tilted frame of the bulk-flow observers. Then, given the pressure-free nature of the cosmic fluid (recall that \( \dot{\varrho} = p = 0 \)), we may write

\[
\tilde{\Delta}_a' = -\tilde{Z}_a + \frac{3aH}{\varrho} (\tilde{q}_a' + 4H\tilde{q}_a) - \frac{a}{\varrho} \tilde{D}_a\tilde{D}^b\tilde{q}_b, \quad (7.2.2)
\]

where \( \tilde{q}_a = -\varrho\tilde{v}_a \) to leading order. Substituting this flux vector into the right-hand side of the above, the latter recasts as [79]

\[
\tilde{\Delta}_a' = -\tilde{Z}_a - 3aH (\tilde{v}_a' + H\tilde{v}_a) + a\tilde{D}_a\tilde{\vartheta}, \quad (7.2.3)
\]

thus relating the time-derivative of the peculiar velocity to inhomogeneities in the matter density and in the universal expansion (monitored by \( \tilde{\Delta}_a \) and \( \tilde{Z}_a \) respectively). Taking the spatial divergence of (7.2.3), using the linear commutation law between temporal and spatial gradients given in Appendix A.2 (see expression (A.8) there), and then solving the resulting expression for \( \tilde{D}^a\tilde{v}_a' \), we arrive at

\[
\tilde{D}^a\tilde{v}_a' = -H\tilde{\vartheta} + \frac{1}{3H} \tilde{D}^2\tilde{\vartheta} - \frac{1}{3a^2H} (\tilde{\Delta}' + \tilde{Z}) \quad , \quad (7.2.4)
\]

37
with $\Delta$ and $\tilde{Z}$ representing (scalar) perturbations in the matter density and the expansion respectively (see § 5.1.1 for the related definitions).

The auxiliary relation (7.2.4) combines with (7.2.1) to provide the linear expression of the peculiar Raychaudhuri equation, namely [79]

$$\ddot{\vartheta} + \frac{1}{3H} \dot{D}^2 \vartheta - \frac{1}{3a^2 H} \left( \dot{\Delta} + \dot{\tilde{Z}} \right).$$

(7.2.5)

According to the above, temporal variations in the density-perturbation spectrum and perturbations in the universal expansion (represented by $\Delta'$ and $\tilde{Z}$ respectively) can force the bulk to expand or contract locally. Put another way, even if $\vartheta = 0$ initially, the presence of $\Delta'$ and $\tilde{Z}$ on the right-hand side of (7.2.5) will lead to $\vartheta' \neq 0$ eventually.

Expression (7.2.5) has a clear scale-dependence encoded in the spatial (covariant) Laplacian term seen on its right-hand side. One can make this dependence explicit by harmonically decomposing the perturbed variables. More specifically, employing the standard scalar harmonic function $Q(n)$ (with $\_Q(n) = 0$ and $D^2 Q(n) = (n/a)^2 Q(n)$), Eq. (7.2.5) recasts into

$$\ddot{\vartheta}_n = -2H\dot{\vartheta}_n + \frac{n^2}{3a^2 H} \vartheta_n - \frac{1}{3a^2 H} \left( \Delta'_n + \tilde{Z}_n \right).$$

(7.2.6)

where $n$ is the comoving wavenumber of the perturbed mode (with $\lambda_n = a/n$ being the physical wavelength).\(^5\) The above shows that the evolution of the peculiar volume scalar ($\dot{\vartheta}$), as well as the strength of its effects, depend on the scale ($\lambda_n$) of the peculiar-velocity perturbation (i.e. of the bulk flow) in question. We will return to the implications of this scale-dependence in the following sections.

### 7.3 Alternative forms of the peculiar Raychaudhuri equation

One can recast Eq. (7.2.5) in a form that includes anisotropies, as well as inhomogeneities. In order to do so, we recall that the spatial gradients in the volume expansion scalar satisfy constraint (3.1.8) – see § 3.1 earlier. Linearised in the tilted frame of the bulk-flow observers, where $\tilde{q}_a = -\rho \tilde{v}_a$, the latter reads

$$2\tilde{Z}_a = 3a \left( \tilde{D}^b \tilde{\sigma}_{ab} - \text{curl} \tilde{\omega}_a - \rho \tilde{v}_a \right),$$

(7.3.1)

with $\text{curl} \tilde{\omega}_a = \tilde{\varepsilon}_{abc} \tilde{D}^b \tilde{\omega}^c$. Taking the spatial divergence of the above, while keeping in mind that $D^a \text{curl} \tilde{\omega}_a = 0$ to first approximation, leads to

$$2\tilde{Z} = 3\tilde{\Sigma} - 9a^2 H^2 \Omega \tilde{\vartheta},$$

(7.3.2)

\(^5\) In deriving (7.2.6) we have also set $\dot{\vartheta} = \sum_n \dot{\vartheta}_n Q(n)$, $\Delta = \sum_n \Delta(n) Q(n)$ and $\tilde{Z} = \sum_n \tilde{Z}(n) Q(n)$. Also note that $D^a \tilde{\vartheta}_n = 0 = D^a \Delta(n) = D^a \tilde{Z}(n)$, $\dot{\vartheta}(n) = 0$ and $D^2 Q(n) = (n/a)^2 Q(n)$. 

38
where $\tilde{Z} = a\tilde{D}^a\tilde{Z}_a$ – see § 5.1.1 – and $\tilde{\Sigma} = a^b\tilde{D}^a\tilde{D}^b\tilde{\sigma}_{ab}$ by definition. Also $\Omega = \rho/3H^2$ is the density parameter of the background universe. Substituting the above auxiliary result in Eq. (7.2.5, we arrive at [79]

$$\tilde{\vartheta}' = -2H \left(1 - \frac{3}{4} \Omega\right) \tilde{\vartheta} + \frac{1}{3H} \tilde{D}^2 \tilde{\vartheta} - \frac{1}{3a^2H} \left(\tilde{\Delta}' + \frac{3}{2} \tilde{\Sigma}\right).$$

(7.3.3)

This version of the peculiar Raychaudhuri equation explicitly incorporates the effects of spatial anisotropy (via the shear scalar $\Sigma$), in addition to those due to inhomogeneity. At the same time, the scale dependence remains unchanged. We should also note that, in closed FRW models with $\Omega > 4/3$, the “friction” effect of the background expansion on $\tilde{\vartheta}'$ is reversed.
8 Relative-motion effects on the deceleration parameter

Relative motions are known to interfere with the way we interpret the observations. In fact, the history of astronomy is rife with examples where relative-motion effects have led to a gross misinterpretation of reality. In this section, we will consider the implications of the observers’ peculiar motion for their measurement of the deceleration parameter.

8.1 Two deceleration parameters

Earlier, in § 7.1 – see Eq. (7.1.1a) there, we saw that \( \ddot{\Theta} \neq \Theta \) at the linear level. Put another way, the expansion rate of universe measured by the real observers differs from the one measured by their fictitious counterparts in the idealised coordinate system of the smooth Hubble flow. Moreover, taking the time derivative of (7.1.1a) and keeping up to first-order terms gives

\[
\dot{\Theta} = \ddot{\Theta} + \ddot{\vartheta}'.
\] (8.1.1)

Therefore, the rate of the universal deceleration/acceleration between the aforementioned two frames differs as well and the difference is entirely due to the real observers’ peculiar motion. This immediately implies that the deceleration parameters measured in the Hubble frame and in the coordinate system of the bulk-flow are also different. Indeed, the deceleration parameters defined in the tilted and in the Hubble frames are

\[
\dot{q} = -\left(1 + \frac{3\ddot{\Theta}'}{\Theta^2}\right) \quad \text{and} \quad q = -\left(1 + \frac{3\ddot{\Theta}}{\Theta^2}\right),
\] (8.1.2)

respectively [28,29]. Solving the above definitions for \( \ddot{\Theta}' \) and \( \ddot{\Theta} \) and substituting the resulting expressions back into Eq. (8.1.1) leads to the following linear relation

\[
1 + \ddot{q} = (1 + q)\left(1 + \frac{\ddot{\vartheta}}{\Theta}\right)^{-2} - \frac{3\ddot{\vartheta}'}{\Theta^2}
= (1 + q)\left(1 + \frac{\ddot{\vartheta}}{\Theta}\right)^{-2} + \left[1 + \frac{1}{2}(1 + 3w)\Omega\right]^{-1}\frac{\ddot{\vartheta}'}{\Theta}. \] (8.1.3)

between the two deceleration parameters. Note that \( \ddot{\Theta} = \Theta + \ddot{\vartheta} \) to first approximation and \( 3\ddot{\Theta} = -\Theta^2[1 + [(1 + 3w)\Omega/2]] \) in the Friedmann background [28,29]. The above result confirms that the deceleration parameters measured in the Hubble and in the tilted frames differ and that their difference is triggered by relative-motion effects alone. Indeed, in the absence of peculiar flows \( \ddot{\vartheta} \) vanishes and \( \ddot{q} = q \). Then, the next question is how big this difference can be and under what conditions.

Expression (8.1.3) holds in an almost-FRW universe with arbitrary curvature that is filled with a single barotropic fluid. Let us now confine to the post-recombination era (dominated
by pressureless dust) and assume spatial flatness as well. In other words, consider a perturbed Einstein-de Sitter universe (with \( p = 0 = w \) at to first order and \( \Omega = 1 \) in the background). Throughout the linear regime \( \dot{\vartheta}/\Theta \ll 1 \) by default, in which case a simple Taylor expansion reduces Eq. (8.1.3) to

\[
1 + \tilde{q} = (1 + q) \left( 1 - \frac{2}{3} \frac{\vartheta}{H} \right) + \frac{1}{2} \frac{\vartheta'}{H},
\]

since \( \Theta = 3H \) to zero order. Because of the weakness of the \( \dot{\vartheta}/H \)-ratio at the linear level, the only appreciable relative-motion effects on \( \tilde{q} \) (if any) come from time-derivative ratio \( \dot{\vartheta}'/H \). More specifically, the effects depend solely on the time derivative of the local expansion/contraction scalar \( (\dot{\vartheta}') \), which was studied in the previous section (see § 7.2 earlier). On these grounds, Eq. (8.1.4) simplifies further to

\[
\tilde{q} = q + \frac{1}{2} \frac{\vartheta'}{H} = q - \frac{1}{3} \frac{\dot{\vartheta}'}{H^2},
\]

since \( \dot{H} = -3H^2/2 \) in an Einstein-de Sitter universe. This linear relation compares the deceleration parameter \( (\tilde{q}) \) measured in the bulk-frame of the real observers, to the one measured in the idealised coordinate system of the Hubble expansion \( (q) \). Following (8.1.5) \( \tilde{q} \neq q \) and their difference is entirely due to relative-motion effects. In what follows, we will attempt to estimate the magnitude (as well as the sign) of the correction term \( (\dot{\vartheta}'/2H = -\dot{\vartheta}'/3H^2) \) seen on the right-hand side of (8.1.5),

\[\text{8.2 The scale-dependence of } \tilde{q} \]

The temporal evolution of the local volume scalar \( (\dot{\vartheta}) \) is monitored by the Raychaudhuri equation of the associated peculiar motion (see § 6.4 and § 7.2 previously). Confining to the linear regime of the post-equipartition universe, expressions (7.2.5) and (8.1.5) combine to

\[
\tilde{q} = q + \frac{2}{3} \frac{\dot{\vartheta}}{H} - \frac{1}{9H^2} \tilde{D}^2 \dot{\vartheta} + \frac{1}{9a^2H^3} \left( \dot{\Delta}' + \tilde{Z} \right),
\]

with an implicit scale-dependence because of the spatial Laplacian term on the right-hand side. The latter becomes explicit by applying a simple Fourier splitting to the perturbed variables. Indeed, setting \( \dot{\vartheta} = \sum_n \dot{\vartheta}_n Q^{(n)}, \Delta = \sum_n \Delta(n) Q^{(n)}, \tilde{Z} = \sum_n \tilde{Z}_n Q^{(n)} \) and \( \tilde{q} = \sum_n \tilde{q}_n Q^{(n)} \) (see also footnote 5), leads to

\[
\tilde{q}_n = q + \frac{2}{3} \left[ 1 + \frac{1}{6} \left( \frac{\lambda_H}{\lambda_n} \right)^2 \right] \frac{\dot{\vartheta}_n}{H} + \frac{1}{9} \left( \frac{\lambda_H}{\lambda_K} \right)^2 \left( \frac{\Delta(n)}{H} + \frac{\tilde{Z}_n}{H} \right),
\]

for the \( n \)-th harmonic mode. In the above, \( \lambda_H = 1/H \) represents the Hubble horizon, \( \lambda_n = a/n \) is the wavelength of the peculiar velocity perturbations (i.e. essentially he scale of the bulk

\footnote{Although the \( \dot{\vartheta}/H \)-ratio is always much smaller than unity during the linear phase, this is not necessarily the case for the ratio of their derivatives \( \dot{\vartheta}'/H \).}
flow) and $\lambda_K = a/|K|$ (with $K = \pm 1$) is the curvature scale of the universe.

Recalling that $(\lambda_H/\lambda_n)^2 = |1 - \Omega|$, with $\Omega \to 1$ in an Einstein-de Sitter model, the last term on the right-hand side of (8.2.2) is negligible for all practical purposes. Then, Eq. (8.2.2) reduces to

$$
\ddot{q}(n) = q + \frac{2}{3} \left[ 1 + \frac{1}{6} \left( \frac{\lambda_H}{\lambda_n} \right)^2 \right] \frac{\ddot{\theta}(n)}{H},
$$

(8.2.3)

with $\ddot{\theta}(n)/H \ll 1$ throughout the linear regime. In the above, $\ddot{q}(n)$ is the value of the deceleration parameter on a scale $\lambda_n$, as measured by observers residing at the centre of bulk flow with local expansion/contraction scalar $\ddot{\theta}(n)$. On the other hand, $q$ is the deceleration parameter measured in the coordinate system of the smooth Hubble flow, namely $\ddot{q}(n)$ is the deceleration parameter of the universe.

Following (8.2.3), the difference between $\ddot{q}(n)$ and $q$ is entirely due to relative-motion effects and also has an explicit scale dependence. More specifically, on scales much larger than the Hubble horizon, namely when $\lambda_H/\lambda_n \ll 1$, the “correction” due to the observers peculiar flow is completely negligible and $\ddot{q}(n) \to q$. This agrees with the general expectation that the impact of peculiar velocities fades away as one moves to progressively larger lengths.

This picture changes drastically when one moves to scales well inside the Hubble horizon, where $\lambda_H/\lambda_n \gg 1$. There, the relative-motion effects are no longer negligible. On those wavelengths, namely much closer to the observer at the centre of the bulk peculiar flow, expression (8.2.3) acquires the form

$$
\ddot{q}(n) = q + \frac{1}{9} \left( \frac{\lambda_H}{\lambda_n} \right)^2 \frac{\ddot{\theta}(n)}{H},
$$

(8.2.4)

where the relative-motion correction can now be strong (even though $\ddot{\theta}(n)/H \ll 1$), since the overall effect is more sensitive to the scale-ratio $(\lambda_H/\lambda_n)$.

Qualitatively speaking, the effect of the observer’s peculiar motion depends on the sign of the local volume scalar $(\ddot{\theta}(n))$. In particular, the deceleration parameter measured by observers inside locally expanding bulk flows (with $\ddot{\theta}(n) > 0$) is larger than its Hubble-flow counterpart. Observers in locally contracting peculiar motions, on the other hand, assign smaller values to their deceleration parameter (i.e. $\ddot{q}(n) < q$ when $\ddot{\theta}(n) < 0$). Quantitatively, the strength of the relative-motion effect is scale-dependent and the smaller the scale the stronger the effect. This is also in accord with our expectation that the impact of the peculiar velocities should fade away of progressively larger wavelengths.

What is most intriguing, is that inside (slightly) contracting bulk flows the value of the deceleration parameter $(\ddot{q}(n))$ can drop below zero, while at the same time the value of $q$ remains negative. In other words, observers inside contracting bulk flows may “experience” accelerated expansion, in an otherwise decelerating universe. Such an effect is only local of course. An artifact of the observers’ motion relative to the universal expansion. Next, we well look into this possibility in more detail.
8.3  The peculiar Jeans length

The relative motion effects start to dominate and thus dictate the local value of the deceleration parameter, when the correction term seen on the right-hand side of (8.2.4) becomes equal to $q$. Following (8.2.4), this happens at a specific length-scale given by [31]

$$\lambda_P = \sqrt{\frac{1}{9q}} \frac{\sqrt{\dot{\gamma}}}{\lambda H}.$$  \hspace{1cm} (8.3.1)

since $\dot{\gamma} \geq 0$ and $q > 0$ in all FRW models with conventional matter. Also note that, hereafter, we will drop the mode index $(n)$ for the economy of the presentation. The above length scale marks the threshold below which the local deceleration/acceleration is no longer determined by the background global expansion, but by peculiar velocity perturbations. In this respect, $\lambda_P$ resembles the Jeans length ($\lambda_J$) familiar from linear perturbation studies (e.g. see [21,22]). Recall that the Jeans length sets the scale below which linear gradients in the pressure (relativistic or thermal) dominate over the background gravitational pull and thus prevent the density perturbations from growing. We will therefore refer to $\lambda_P$ as the "peculiar Jeans length" to underline the analogies between these two characteristic scales [31].

A comparison between $\lambda_P$ and $\lambda_J$ is due here. In accord with standard linear cosmological perturbation theory, the Jeans length is given by (e.g. see [21,22])

$$\lambda_J \propto c_s \lambda H,$$  \hspace{1cm} (8.3.2)

with $c_s = \sqrt{dp/d\rho}$ representing the associated "sound speed". Note that $c_s$ is measured in units of the speed of light, which has been normalised to $c = 1$. This means that $c_s = 1/\sqrt{3}$ for relativistic species and $c_s \ll 1$ for low-energy matter. Also, the proportionality factor in (8.3.2) depends on the equation of state of the cosmic medium and is of order unity (e.g. see [21,22]). The close analogy between expressions (8.3.1) and (8.3.2) is clear, with the dimensionless ratio $|\dot{\gamma}|/H$ playing for the peculiar Jeans length the role $c_s$ plays for the Jeans length proper.

8.4  The transition scale

To demonstrate the pivotal role of $\lambda_P$ and its significance, when studying the effects large-scale peculiar flows on the deceleration parameter of the universe, we substitute definition (8.3.1) into Eq. (8.2.4). The latter then recasts into [27]

$$\ddot{q} = \ddot{q}^\pm = q \left[ 1 \pm \left( \frac{\lambda_P}{\lambda} \right)^2 \right],$$  \hspace{1cm} (8.4.1)

with the plus/minus sign corresponding to locally expanding/contracting bulk peculiar motions (with $\dot{\gamma} \geq 0$ respectively). Consequently, the local deceleration parameter ($\ddot{q}^+$) measured
by observers inside expanding bulk flows is larger than \( q \). On the other hand, the situation is reversed when the observers reside within contracting bulk peculiar motions. There, expression (8.4.1) ensures that \( \tilde{q}^- \lesssim q \). Nevertheless, on scales larger than \( \lambda_P \) the correction term due to the observers relative motion is negligible and \( \tilde{q}^\pm \rightarrow q \) as expected. When \( \lambda \ll \lambda_P \), however, the situation changes drastically. On these wavelengths, Eq. (8.4.1) leads to the following two alternatives [31,27]

\[
\tilde{q} > 2q, \quad \text{when} \quad \tilde{\vartheta} > 0, \quad (8.4.2)
\]

or

\[
\tilde{q} < 0, \quad \text{when} \quad \tilde{\vartheta} < 0. \quad (8.4.3)
\]

Therefore, observers inside (slightly) expanding bulk flows will measure a deceleration parameter twice as large as that of the actual universe on scales smaller than the associated peculiar Jeans length. In contrast, their counterparts residing in (slightly) contracting bulk peculiar motions will assign negative values to their deceleration parameter. To the former group of observers the universe will seem over-decelerated, whereas to the latter it will appear under-decelerated, or even accelerated in some cases. Nevertheless, the over-deceleration/acceleration is not real, but a local apparent effect triggered by the observers peculiar motion relative to the universal expansion.

It follows that, when dealing with contracting bulk flows, the peculiar Jeans length \( \lambda_P \) also marks the “transition scale” \( \lambda_T \), where the value of the locally measured deceleration parameter turns from positive to negative. We may therefore define

\[
\lambda_T = \sqrt{\frac{1}{9q} \frac{|\tilde{\vartheta}|}{H^2}} \lambda_H, \quad \text{when} \quad \tilde{\vartheta} < 0, \quad (8.4.4)
\]

as the transition length in locally contracting bulk peculiar motions. Measurements made within \( \lambda_T \) will lead to (apparent) negative values for the deceleration parameter, solely due to the observer’s peculiar flow. Then, provided that \( \lambda_T \) is large enough, namely of cosmological relevance (i.e. few hundred Mpc), an unsuspecting observer may misinterpret the local change in the sign of the deceleration parameter as recent global acceleration. Another way of putting it, is by saying the observers have misinterpreted their local contraction as global acceleration.

The evolution of the deceleration parameter, in terms of scale, follows from Eq. (8.4.1). Assuming that the background universe is an Einstein-de Sitter model (with \( q = 1/2 \)), we find that \( \tilde{q}^\pm \rightarrow 1/2 \) far away from the observer, that is on scales beyond the associated peculiar Jeans length. Closer to the observer, on the other hand, the local deceleration parameter begins to diverge from its background value, turning increasingly larger/smaller than 1/2 for observes residing in expanding/contracting bulk flows. More specifically, at the peculiar Jeans length, the local value of \( \tilde{q}^+ \) exceeds unity (i.e. \( \tilde{q}^+ > 2q = 1 \)), whereas that of \( \tilde{q}^- \) drops below zero (see Fig. 3 and also [27]).

It should be noted that the dashed and the solid curves depicted in Fig. 3 correspond to the simplest form of Eq. (8.4.1), where \( \tilde{\vartheta} \) (and therefore \( \lambda_T \)) remains constant within the bulk-flow domain and does not vary with scale. In a more realistic scenario, however, the local expansion/contraction scalar will most likely have some scale dependence (i.e. we expect that
The scale-distribution of the deceleration parameter ($\dot{q}(\pm)$) in the rest-frame of a bulk flow with peculiar Jeans length $\lambda_P$ (see Eq. (8.4.1)). On large scales the local deceleration parameter approaches that of the background Einstein-de Sitter universe (i.e. $\dot{q}(\pm) \rightarrow q = 1/2$), but on smaller scales it diverges. In (slightly) expanding peculiar flows (dashed curve), $\dot{q}^+ > 1/2$, with the deceleration parameter crossing the $\dot{q}^+ = 1$ mark at $\lambda = \lambda_P$. Inside contracting bulk peculiar motions (solid curve), on the other hand, $\dot{q}^- < 1/2$. There, the local deceleration parameter drops below the $\dot{q}^- = 0$ threshold at the transition scale, namely at $\lambda_T$. The dotted vertical line indicates the nonlinear cutoff scale, below which our linear analysis no longer holds [27].

Nevertheless, even with this simple evolution law, the profile of the solid curve in Fig. 3 naturally reproduces the desired features of the deceleration parameter, namely the value of $\dot{q}$ approaches the Einstein-de Sitter limit ($q = 0.5$) on large scales, while it becomes increasingly negative closer to the observer. Also, the profile of the aforementioned solid curve resembles those of the deceleration parameters reconstructed by the supernovae data [80–83]. There, however, the desired shape was achieved after introducing a two-parameter ansatz for the deceleration parameter. Here, instead, the profile of the solid curve in Fig 3 emerges naturally from the relative-motion effects of the observer’s peculiar flow.

### 8.5 Estimating $\dot{q}^\pm$ and $\lambda_P$

On sub-horizon scales, the local value of the deceleration parameter is given by (8.1.5) in § 8.2. According to the latter expression, $\dot{q}$ depends on the sign and the magnitude of $\vartheta$, as measured at some scale $\lambda$. We will therefore turn to the observations and substitute into (8.1.5) the reported peculiar velocities and scales. These typically measure velocities of the order of few hundred km/sec on scales of few hundred Mpc [8,9,12–15]. Having said that, the surveys provide the mean bulk velocity and not its divergence. The reason is systematic uncertainties interfering the measurements of these higher moments, since they are derivatives of the noisy peculiar-velocity field. Practically speaking, noise-related problems still make the measured values of these moments uncertain (e.g. see [13]).

Here, in an attempt to address the lack of direct data, we will employ standard dimensional-analysis arguments to provide an approximate relation between $\vartheta$ and the measured mean bulk velocity ($\langle \dot{v} \rangle$). In particular, given that the spatial curvature is negligible well inside the
Hubble horizon, we will set
\[ \tilde{\vartheta} \simeq \partial_{\alpha} \tilde{v}^{\alpha} \simeq \pm \frac{\sqrt{3} \langle \tilde{v} \rangle}{\lambda}, \]  
with the plus/minus sign corresponding to expanding/contracting peculiar motions. Then, recalling that \( v_H = \lambda H \) is the Hubble velocity on the at the associated scale, Eqs. (8.2.4) and (8.3.1) become

\[ \tilde{q}^{\pm} \simeq q \pm \frac{\sqrt{3}}{9} \frac{\langle \tilde{\vartheta} \rangle}{\langle \tilde{v} \rangle} \hat{v}_H \]  
and

\[ \lambda_P \simeq \sqrt{\frac{\sqrt{3} \langle \tilde{v} \rangle}{9 q \hat{v}_H \lambda_H}}, \]

respectively. We may now estimate the values of \( \tilde{q}^{(\pm)} \) and \( \lambda_P \) measured inside some of the reported bulk flows (see Table 1). Typically, these surveys claim velocities from 200 to 400 km/sec over regions varying between 150 to 300 Mpc in diameter (e.g. see [8,9,12–15]).

Following Eq. (8.5.2), the numerical estimates of \( \tilde{q}^{\pm} \) quoted in Table 1 depend on the dimensionless ratios \( \tilde{\vartheta}/H \) and \( (\lambda_H/\lambda)^2 \). This makes \( \tilde{q}^{\pm} \) more sensitive to the scale-ratio \( (\lambda_H/\lambda) \), rather than the ratio \( (\hat{\vartheta}/H) \) of the volume scalars. \(^{8}\) According to definition (8.5.3), on the other hand, the peculiar Jeans length \( (\lambda_P) \) is solely determined by the \( \tilde{\vartheta}/H \)-ratio. Finally, we should also note that both \( \tilde{q}^{\pm} \) and \( \lambda_P \) have an additional dependence on the current value of the Hubble parameter, with the relative motion effects getting stronger if the latter was to decrease (and vice versa). \(^{9}\)

### 8.6 Apparent acceleration due to relative-motion effects

Assuming that the reported bulk flows are (slightly) expanding, the deceleration parameters measured at the corresponding scales are the range \(+1.01 \lesssim \tilde{q}^{+} \lesssim +7.08 \) (3rd column in

\(^{7}\) Ignoring spatial curvature implies \( \tilde{\vartheta} \simeq \partial_{x} \tilde{v}^{x} + \partial_{y} \tilde{v}^{y} + \partial_{z} \tilde{v}^{z} \). Then, assuming that \( \tilde{v}^{x} \simeq \tilde{v}^{y} \simeq \tilde{v}^{z} \simeq \tilde{v} \), we may set \( \tilde{\vartheta} \simeq \pm \tilde{v}^{x}/\lambda \) and \( \langle \tilde{\vartheta} \rangle \simeq \sqrt{3} \tilde{v} \), all of which combine to give \( \tilde{\vartheta} \simeq \pm \sqrt{3} \langle \tilde{v} \rangle/\lambda \). \(^{8}\) In Table 1 we can see the impact of the peculiar motion to decrease with increasing scale and vice versa. The weakest effect corresponds to the survey by Nusser & Davis [14], where the deceleration parameter becomes marginally negative \( (\tilde{q}^{(\pm)} \simeq -0.01) \) and the transition scale just exceeds that of the reported bulk flow \( (\lambda_P \simeq 282 \text{ Mpc} \gtrsim \lambda \simeq 280 \text{ Mpc}) \). At the opposite end is the survey of Feldmann et al [13], where the relative-motion effect on both \( \tilde{q} \) and \( \lambda_P \) is quite strong. \(^{9}\) The values of \( \tilde{\vartheta} \) were obtained from the mean bulk-velocity, by means of dimensional-analysis arguments (see footnote 7). The estimates of \( \tilde{\vartheta}/H \) used in Table 1 may therefore change when direct measurements of \( \tilde{\vartheta} \) become available. To compensate for a possible overestimation of \( \tilde{\vartheta} \), we have used the diameter rather then the radius of the reported bulk flows, which reduces the relative-motion effects in Table 1. Alternatively, one could account for the present ambiguity by writing \( \tilde{\vartheta} \simeq \pm \alpha \sqrt{3} \langle \tilde{v} \rangle/\lambda \), with \( 0 < \alpha < 1 \), to avoid overestimating the impact of the peculiar flow on the local deceleration parameter [30]. Then, setting \( \alpha = 1/2 \) reduces the impact on \( \tilde{q} \) by half, assuming that \( \alpha = 1/4 \) the effect drops by another half and so on.
Table 1
Estimates of the local deceleration parameter ($\tilde{q}^\pm$ – see Eq. (8.5.2)) measured by observers residing inside bulk flows, with scales and velocity measurements like those reported in [8,9,12–15]. Note that $\tilde{q}^+$ corresponds to slightly expanding and $\tilde{q}^-$ to slightly contracting bulk flows. In the latter case the peculiar Jeans length ($\lambda_P$ – see last column) also marks the transition scale ($\lambda_T$), where $\tilde{q}^-$ turns negative (see Eq. (8.4.4) and also Fig. 4). In all cases, the universe is assumed to decelerate globally with $q = +0.5$ relative to the Hubble/CMB frame. Finally, our numerical estimates correspond to $H \simeq 70\text{ km/sec Mpc}$ and $\lambda_H = 1/H \simeq 4 \times 10^3\text{ Mpc}$ today.

<table>
<thead>
<tr>
<th>Survey</th>
<th>$\lambda$ (Mpc)</th>
<th>$\langle \tilde{v} \rangle$ (km/sec)</th>
<th>$\tilde{q}^+$</th>
<th>$\tilde{q}^-$</th>
<th>$\lambda_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nusser &amp; Davis (2011)</td>
<td>280</td>
<td>260</td>
<td>+1.01</td>
<td>-0.01</td>
<td>282</td>
</tr>
<tr>
<td>Scrimgeour, et al (2016)</td>
<td>200</td>
<td>240</td>
<td>+1.81</td>
<td>-0.81</td>
<td>323</td>
</tr>
<tr>
<td>Ma &amp; Pan (2014)</td>
<td>170</td>
<td>290</td>
<td>+3.05</td>
<td>-2.05</td>
<td>384</td>
</tr>
</tbody>
</table>

Table 1). When the bulk flows reported in [8,9,12–15] are (slightly) contracting, on the other hand, the situation changes drastically. In that case, Eq. (8.5.2) implies negative values for the local deceleration parameter, with $-6.08 \lesssim \tilde{q}^- \lesssim -0.01$ (see 4th column in Table 1). Also, the peculiar Jeans length ($\lambda_P$) now defines the transition scale ($\lambda_T$), where the deceleration parameter crosses the $\tilde{q} = 0$ divide and turns from positive to negative. In the surveys quoted in Table 1, the peculiar Jeans length ($\lambda_P$), which coincides with the transition scale ($\lambda_T$) in contracting bulk flows, varies between 280 and 500 Mpc (see 5th column in Table 1).

It should be emphasised that, in all of the aforementioned cases, the universe is assumed to decelerate globally, with $q = +0.5$ relative to the Hubble/CMB frame. This means that neither the over-deceleration nor the acceleration seen in Table 1 are real, but local artifacts of the observers’ relative motion. Nevertheless, the affected scales are large enough, since $\lambda_P$ is of the order of few hundred Mpc, to create the false impression that there was a recent global change in the expansion rate of the universe.

Before closing this section, we should point out that the negative values of the deceleration parameter, quoted in the 4th column of Table 1, have been obtained within a perturbed Einstein-de Sitter cosmology filled with low-energy dust. There was no need to violate the strong-energy condition, to introduce a cosmological constant, or to modify general relativity in any way. The inferred accelerated expansion is not real, since the universe is still globally decelerating, but a local artifact of the observers’ peculiar motion relative to the smooth universal (Hubble) flow. Nevertheless, the affected scales are large enough (the transition scale is of the order of few hundred Mpc (see 5th column in Table 1) to mislead the unsuspecting observers, who might then misinterpret their local contraction as global acceleration (see Fig. 4 for a schematic description).
Fig. 4. Observer ($O$) inside a bulk flow (central white section) like those quoted in Table 1, with 4-velocity $\tilde{u}_a$ and peculiar velocity $\tilde{v}_a$, relative to the smooth Hubble expansion (identified with the $u_a$-field). Around the observer there is a spherical region (shaded area), with size determined by the associated peculiar Jeans length ($\lambda_P$) and corresponding to redshift $z_{\lambda_P}$. In the case of contracting bulk flows (with $\dot{\vartheta} < 0$), the outer limits of the shaded domain mark the transition scale ($\lambda_T$), where the locally measured value of the deceleration parameter ($\tilde{q}^-$) turns from positive to negative (see Table 1). From the observer’s point of view, the value of $\tilde{q}^-$ becomes progressively less negative away from them, it crosses the $\tilde{q}^- = 0$ threshold at the transition scale and starts taking positive values beyond $\lambda_T$, eventually approaching $q = 1/2$ on large enough lengths (see Fig. 3). This can mislead the unsuspecting observer to believe that their universe started to accelerate at $z = z_{\lambda_T}$ and in so doing erroneously interpret the local change in the sign of $\tilde{q}^-$ as recent global acceleration.

8.7 Signatures of the bulk-flow scenario

One might wonder whether there is a way for the unaware bulk-flow observers to realise that they have been experiencing a mere illusion. The answer should be in the data, which must contain the “imprints” of the relative-motion effects. A first sign could come from the scale-distribution of the deceleration parameter, which should become progressively more negative on smaller wavelengths (i.e. closer to the observer). In addition, the profile of $\tilde{q}^-$-distribution should agree qualitatively with that of the solid curve depicted in Fig. 3. Recall that the shape of the latter reflects the scale dependence (i.e. the $1/\lambda^2$-law) seen in Eq. (8.4.1). This corresponds to the simplest scenario, where $\tilde{v}^-$ remains constant inside the baulk-flow domain.

A second signature of the bulk-flow scenario in the data should be an apparent (Doppler-like) dipolar anisotropy in the sky-distribution of the deceleration parameter. The value of $\tilde{q}$ should be smaller than the average towards a certain point in the sky and equally larger in the opposite. In other words, the universe should appear to accelerate faster in one direction and equally slower in the antipodal. For typical bulk-flow observers, with velocity close to the mean, the aforementioned dipole-like anisotropy should be weak [29]. Moreover, provided the CMB dipole seen by these observers is also due to the relative motion, the two axes should not lie fairly close. For atypical bulk-flow observers, however, with individual peculiar velocities considerably different than the mean, the apparent dipolar anisotropy could be large. The possibility that we might be such atypical observers cannot be excluded [84].

Relative motions are typically associated with apparent (Doppler-like) dipolar anisotropies.
On these grounds the presence of a dipole in the sky-distribution of the deceleration parameter, due to the observer’s peculiar flow, is intuitively plausible. The possibility for such an anisotropy was first suggested in [28] and then theoretically demonstrated in [29], by taking into account the (apparent) shear-like effects of the relative motion. Recently, an alternative theoretical approach, which employed the null geodesics of the radiation signals to analyse the impact of relative motion on the luminosity distance, reached an analogous conclusion [85].

In addition to the theoretical predictions, there have also been observational reports claiming that a dipolar axis (close to that of the CMB) may exist in the supernovae data [32–37]. Put another way, our universe may indeed seem to accelerate faster towards one direction in the sky and equally slower in the opposite. Nevertheless, the aforementioned early reports did not relate the detected axis to relative-motion effects. This connection was made recently, after re-examining the Joint Lightcurve Analysis (JLA) data of type Ia supernovae, in [38]. There, the authors detected a fairly strong dipole in the sky-distribution of the deceleration parameter that was also closely aligned with that of the CMB. Interestingly, the statistical significance of the $q$-monopole dropped, increasing the chance that the inferred universal acceleration may actually be an artefact of our peculiar motion. Future observations, with more and better data, should help clarify the $q$-dipole debate [86–88].

Finally, it is worth mentioning that relative motion-effects can induce an apparent (also Doppler-like) dipole in the sky-distribution of the Hubble parameter. The physical principle/mechanism is essentially identical to the one triggering $\hat{q}$-dipole discussed here. Therefore, observers living inside large-scale bulk flows may also see their universe expanding faster towards a particular direction in the sky and equally slower in the opposite. Claims that such a Hubble-dipole may actually exist in the data were recently made in [89,90].
Newtonian physics treats time and space as independent and absolute entities. In relativity, on the other hand, time and space are closely interconnected, while observers moving with respect to each other measure their own time and space. Moreover, Newton’s gravity is a force coming from a potential, whereas in Einstein’s theory gravity is the manifestation of spacetime curvature. In this section, we will examine whether and to what extent such fundamental differences can modify the results discussed so far.

9.1 The Newtonian covariant approach

As mentioned at the beginning of § 2, the covariant approach to fluid dynamics was originally formulated within the Newtonian framework [43,44], before extended to relativity. Next, we will go through the basics of the formalism, referring the reader to [46] for further discussion and a comparison to the relativistic version.

Consider a group of observers moving with velocity \( u_\alpha \) in a 3-dimensional Euclidean space. Suppose also that \( h_{\alpha\beta} \) is the associated metric tensor, relative to a coordinate system \( \{x^\alpha\} \). Then, \( h_{\alpha\alpha} = 3 \) by construction and \( v^2 = h_{\alpha\beta}v^\alpha v^\beta \) for any vector field \( v_\alpha \). In a Cartesian frame, we have \( h_{\alpha\beta} = \delta_{\alpha\beta} \), where \( \delta_{\alpha\beta} \) is the Kronecker delta. Otherwise, \( h_{\alpha\beta} \neq \delta_{\alpha\beta} \) and both \( h_{\alpha\beta} \) and \( h^{\alpha\beta} \) (with \( h_{\alpha\mu}h^{\mu\beta} = \delta_\alpha^\beta \)) are needed to raise and lower indices. In that case, one also needs to use “covariant” rather than partial derivatives to compensate for the “curvature” of the coordinate system [46,91]. Here, we will keep using upper and lower indices, even when dealing with Cartesian vectors and tensors, to facilitate the comparison between the Newtonian and the relativistic formulae. One can distinguish the Newtonian equations from the use of ordinary partial (instead of covariant) derivatives and from the Greek (rather than Latin) indices.

Also, the Newtonian alternating tensor \( (\varepsilon_{\alpha\beta\mu} = \varepsilon_{[\alpha\beta\mu]} \), with \( \varepsilon_{123} = 1 \) is purely spatial by construction and satisfies the constraint

\[
\varepsilon_{\alpha\beta\mu}\varepsilon^{\alpha\beta\kappa} = 3!\delta_{[\alpha}^{\nu}\delta_{\beta}^\gamma\delta_{\mu]}^\kappa,
\]

so that \( \varepsilon_{\alpha\beta\mu}\varepsilon^{\alpha\beta\kappa} = 2\delta_{[\beta}^\gamma\delta_{\mu]}^\kappa \), \( \varepsilon_{\alpha\beta\mu}\varepsilon^{\alpha\kappa\mu} = 2\delta_\mu^\kappa \) and \( \varepsilon_{\alpha\beta\mu}\varepsilon^{\alpha\beta\mu} = 6 \).

9.2 Newtonian hydrodynamics

The Newtonian description of the covariant kinematics is closely analogous to that of its relativistic 1+3 counterpart. Focusing on cosmology, we consider an expanding universe and introduce a family of observers moving along with the cosmic fluid.
9.2.1 The gravitational field

In Newton’s theory gravity is a force that derives from the associated potential ($\Phi$). The latter couples to the density of the matter via Poisson’s equations, namely

$$\partial^2 \Phi = \frac{1}{2} \rho,$$

(9.2.1)

where $\partial^2 = \partial^\alpha \partial_\alpha$ is the Euclidean Laplacian operator. Note that, in contrast to general relativity, only the density of the matter contributes to the Newtonian gravitational field. There is no input from the pressure of from any flux that may be present (see § 2.2 for a comparison). The latter difference between the two theories will prove crucial, when dealing with peculiar motions at the Newtonian limit (see § 10 below).

Proceeding along the steps of the relativistic treatment (see § 3.1 earlier), we introduce the 3-velocity field ($u_\alpha$) to monitor the motion of our observers. Relative to the $u_\alpha$-field, the time derivative of a general (tensorial) quantity ($T$) is the convective derivative $\mathbf{T} = \partial_t T + u^\alpha \partial_\alpha T$. The (inertial) acceleration of the matter, for example, is given by $\ddot{u}_\alpha = \partial_t u_\alpha + u^\beta \partial_\beta u_\alpha$. The latter adds to the gravitational acceleration, giving [46,91]

$$A_\alpha = \ddot{u}_\alpha + \partial_\alpha \Phi,$$

(9.2.2)

which provides the total force acting on a fluid element via the Euler, or the Navier-Stokes, equation. Note that the 3-vector $A_\alpha$ defined above is the Newtonian analogue of the 4-acceleration vector introduced in § and vanishes when the fluid element is moving under gravity and/or inertia alone (see § 9.2.3 below).

9.2.2 Kinematics

Further kinematic information is obtained by splitting the gradient of the velocity vector. Given that time and space are independent physical entities in Newtonian physics, namely in the absence of spacetime continuum, the gradient of the latter decomposes as [46,91]

$$\partial_\beta u_\alpha = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta},$$

(9.2.3)

with no acceleration vector on the right-hand side (compare to Eq. (3.1.1) in § 3.1). In direct analogy to the relativistic study, $\Theta = \partial^\alpha u_\alpha$ is the volume scalar, $\sigma_{\alpha\beta} = \partial_\beta u_\alpha$ is the shear tensor and $\omega_{\alpha\beta} = \partial_\beta u_\alpha$ is the vorticity tensor associated with the $u_\alpha$-field. As before, the former implies expansion/contraction (when positive/negative respectively), while nonzero shear and vorticity ensure the presence of kinematic anisotropies and rotation. Also, the volume scalar defines the cosmological scale factor ($a$), since $\dot{a}/a = \Theta/3$ by construction, which means that $\Theta = 3H$ in a Friedmann model.

Just like their relativistic counterparts, the Newtonian kinematic variables satisfy a set of three propagation equations supplemented by an equal number of constrains. The are obtained by
the Newtonian analogues of the Ricci identities, which assume the “trivial” form
\[ \partial_t \partial_{\beta} u_\alpha = 0 \quad \text{and} \quad \partial_{\beta} \partial_{\mu} u_\alpha = 0, \quad (9.2.4) \]
for any vector \( u_\alpha \). The former of the above leads to the propagations formulae and the latter to the supplementary constrains. In particular, following [92], the gradient of definition (9.2.2) and decomposition (9.2.3) couple with (9.2.4a) to give
\[
(\partial_\beta u_\alpha) = -\frac{1}{9} \Theta^2 \delta_{\alpha\beta} - \frac{2}{3} \Theta (\sigma_{\alpha\beta} + \omega_{\alpha\beta}) - \partial_\beta \partial_{\alpha} \Phi - \partial_\beta A_\alpha - \sigma_{\mu\alpha} \sigma^\mu_{\gamma\beta} + \omega_{\mu\alpha} \omega^\mu_{\beta} - 2 \sigma_{\mu(\alpha} \omega^\mu_{\beta)}. \quad (9.2.5)
\]
The above contains collective information on the kinematics of the Newtonian fluid, which can be decoded by isolating its trace, its symmetric traceless part and finally its antisymmetric component. More specifically, the trace of (9.2.5) leads to
\[
\Theta = -\frac{1}{3} \Theta^2 - \frac{1}{2} \rho + \partial^\alpha A_\alpha - 2 \left( \sigma^2 - \omega^2 \right), \quad (9.2.6)
\]
which is the Newtonian version of the Raychaudhuri equation. Comparing the above to the relativistic formula (see expression (3.1.3) in § 3.1), one notices that there is no pressure contribution to the gravitational field (see Eq. (10.5.1) above and related discussion) and also the absence of a \( A_\alpha A^\alpha \)-term on the right-hand side of (9.2.6).

Proceeding along the lines of [92], the symmetric trace-free and the skew part of (9.2.5) provide the Newtonian evolution formulae for the shear and vorticity, reading
\[
\dot{\sigma}_{\alpha\beta} = -\frac{2}{3} \Theta \sigma_{\alpha\beta} - E_{\alpha\beta} + \partial_{(\alpha} A_{\beta)} - \sigma_{\mu(\alpha} \sigma^\mu_{\beta)} + \omega_{\mu(\alpha} \omega^\mu_{\beta)} \quad (9.2.7)
\]
and
\[
\dot{\omega}_\alpha = -\frac{2}{3} \Theta \omega_\alpha - \frac{1}{2} \text{curl} A_\alpha + \sigma_{\alpha\beta} \omega^\beta, \quad (9.2.8)
\]
respectively. Note that the symmetric and traceless gradient \( E_{\alpha\beta} = \partial_{(\alpha} \partial_{\beta)} \Phi \) monitors the tidal part of the Newtonian gravitational field and corresponds to the relativistic electric Weyl tensor (see definition (2.2.6a) in § 2.2). In addition, \( \omega_\alpha = \epsilon_{\alpha\beta\mu} \omega^\beta \) is the vorticity vector associated with the \( u_\alpha \)-field and \( \text{curl} A_\alpha = \epsilon_{\alpha\beta\mu} \partial^\beta A^\mu \) by definition. Comparing (9.2.7) and (9.2.8) to their relativistic analogues (see Eqs. (3.1.4) and (3.1.5) in § 3.1), we notice the absence of the \( A_{(\alpha} A_{\beta)} \) and \( \pi_{\alpha\beta} \) terms from the right-hand side of (9.2.7). Expression (9.2.8), on the other hand is formally identical to Eq. (3.1.5).

To complete the kinematic description of a Newtonian medium, one needs to supplement Eqs. (9.2.6)-(9.2.8) with a equal number of constrains. Following [92], we first contract (9.2.4b) with \( \epsilon_{\alpha\beta\mu} \) to obtain the auxiliary relation
\[
\epsilon_{\mu\nu\alpha} \partial^\mu \sigma^\nu_{\beta} + \partial_\beta \omega_\alpha - \partial^\mu \omega_\mu \delta_{\alpha\beta} - \frac{1}{3} \epsilon_{\alpha\beta\mu} \partial^\mu \Theta. \quad (9.2.9)
\]
Then, isolating the trace, the symmetric traceless and the antisymmetric parts of the above, we respectively arrive at [92]

$$\partial^\alpha \omega_\alpha = 0, \quad \text{curl}\sigma_{\alpha\beta} + \partial_{(\alpha} \omega_{\beta)} = 0$$

(9.2.10)

and

$$\frac{2}{3} \partial_\alpha \Theta - \partial^\beta \sigma_{\alpha\beta} + \text{curl}\omega_\alpha,$$

(9.2.11)

where \(\text{curl}\sigma_{\alpha\beta} = \varepsilon_{\mu\nu(\alpha} \partial^{\mu} \sigma_{\nu)\beta}\) by definition. Comparing (9.2.10a)-(9.2.11) to their relativistic analogues (see expressions (3.1.6)-(3.1.8) in §3.1), one immediately notices a number of differences between the two sets, reflecting the different way the two theories address issues as fundamental as space, time and gravity. Among others, in general relativity, the spatial divergence of the vorticity vector is no longer zero (i.e. \(D^\alpha \omega_\alpha \neq 0\) in contrast to (9.2.10a) above).

9.2.3 Conservation laws

In Newtonian gravity, the potential \(\Phi\) couples only to the density of the matter \(\rho\) via Poisson’s formula (see Eq. (10.5.1) in §9.2.1 previously). Also, in Newton’s theory, gravity is a force and the gravitational acceleration is determined by the gradient of the potential. Added to the inertial acceleration, the latter provides the total acceleration \(A_\alpha\) of the motion (see (9.2.2) in §9.2.1), which satisfies the Navier-Stokes equation, namely

$$\rho A_\alpha = -\partial_\alpha p - \partial^\beta \pi_{\alpha\beta},$$

(9.2.12)

where \(p\) and \(\pi_{\alpha\beta}\) (with \(\pi_{\alpha\beta} = \pi_{\beta\alpha}\) and \(\pi^{\alpha}{}_{\alpha} = 0\)) are respectively the (Newtonian/thermal) isotropic pressure and the viscosity of the fluid [46,91]. The above relation ensures that the acceleration vector vanishes when matter moves under inertia or/and gravity alone and therefore it corresponds to the relativistic 4-acceleration (see §3.1 earlier). Comparing (9.2.12) to its relativistic counterpart (i.e. to the momentum-density conservation law (3.2.3) in §3.2), we notice the absence of four terms, three of which manifest the presence of a nonzero energy-flux vector.

Analogous differences also emerge between the Newtonian and the relativistic versions of the continuity equation. Indeed, according to Einstein’s theory the energy-density conservation law is given by Eq. (3.2.2) in §3.2. In Newtonian physics, on the other hand, we have

$$\dot{\rho} = -\Theta \rho.$$  

(9.2.13)

Again, the comparison reveals that a number of terms are “missing” from the right-hand side of the above given Newtonian expression. As in the case of the Navier-Stokes equation, a number of the missing terms are due to a nonzero energy flux. The absence of these latter terms will prove crucial when studying peculiar motions, because then there is an (effective) energy-flux vector by default (see §10 below).
9.3 Inhomogeneous cosmologies

At the Newtonian limit, due to the absence of spacetime curvature, inhomogeneities in the spatial distribution of the physical variables are monitored by their ordinary gradients, rather than their projected covariant derivatives (compare to § 5 earlier). More specifically, the key variables are

\[ \Delta_a = \frac{1}{\rho} \partial_a \rho \quad \text{and} \quad Z_a = a \partial_a \Theta, \tag{9.3.1} \]

instead of their relativistic analogues respectively defined in (5.1.1) and (5.1.2) respectively. Nevertheless, just like their relativistic counterparts, the Newtonian gradients defined above describe inhomogeneities in the density distribution of the matter and in the universal expansion respectively.

Differentiating (9.3.1a) and (9.3.1) with respect to time leads to the nonlinear propagation formulae, monitoring the evolution of the density gradients

\[ \dot{\Delta}_a = -Z_a - (\sigma_{\beta a} + \omega_{\beta a}) \Delta^\beta \tag{9.3.2} \]

and of those in the volume expansion

\[ \dot{Z}_a = -\frac{2}{3} \Theta Z_a - \frac{1}{2} \rho \Delta_a + a \partial_a \partial^\beta A_\beta - 2a \partial_a \left( \sigma^2 - \omega^2 \right) - (\sigma_{\beta a} + \omega_{\beta a}) Z^\beta, \tag{9.3.3} \]

with the overdots indicating convective derivatives along the observers’ motion (e.g. \( \dot{\Delta}_a = \partial_t \Delta_a + u^\beta \partial_\beta \Delta_a \) – see also § 9.2.1 previously). Note that in deriving expressions (9.3.2) and (9.3.3), we have assumed that matter is in the form of a general imperfect fluid. In addition, we have used the Newtonian version of the continuity and the Raychaudhuri equations (see (9.2.13), (9.2.6) and also [91]).

The difference between the (9.3.2), (9.3.3) and their relativistic analogues (compare the above to Eqs. (5.2.1) and (5.2.2) respectively) is profound. Among others, despite our imperfect-fluid assumption, the Newtonian continuity equation does not contain any flux terms. The absence of the latter reflects the fact that, in contrast to general relativity, in Newton’s theory the flux does not contribute to the gravitational field.

Adopting as our background cosmology the Newtonian version of the Friedmann universe, which coincides with the Einstein-de Sitter model for all practical purposes, expressions (9.3.2) and (9.3.3) linearise to

\[ \dot{\Delta}_a = -Z_a \tag{9.3.4} \]

and

\[ \dot{Z}_a = -\frac{2}{3} \Theta Z_a - \frac{1}{2} \rho \Delta_a + a \partial_a \partial^\beta A_\beta, \tag{9.3.5} \]

respectively. Once more, the comparison with the relativistic linear expressions (see expressions (5.3.9) and (5.3.10) in §) reveals profound differences between the two sets. More specifically, the absence of any flux related terms from Eq. (9.3.4) will prove crucial when studying the
(linear) effects of peculiar motions on the deceleration parameter of the universe (see § 10 next).

Finally, we note that setting $A_\alpha = 0$, which means ignoring the (small-scale) role of the thermal pressure (see Eq. (9.2.12) in § 9.2.3 earlier), allows one to solve the system of (9.3.4) and (9.3.5) analytically [91]. The solution shows that $\Delta_\alpha \propto t^{2/3}$ and $Z_\alpha \propto t^{-1/3}$, exactly as its relativistic analogue (see (5.4.2) in § 5.4 previously)
Newton’s theory treats space and time as separate and absolute entities, which are the same for all observers irrespective of their motion. In relativity this is no longer the case. There, relatively moving observers “measure” their own space and time. In what follows, we will apply these fundamentally different approaches to cosmological peculiar motions.

10.1 Galilean transformation

Consider a Newtonian cosmological model and allow for two families of observers moving relative to each other. Let us also identify one group with the (idealised/fictitious) observers following the Hubble flow and the other with those living in typical galaxies, like the Milky Way, drifting with respect to the smooth universal expansion. As the relativistic study before, the coordinate systems associated with these two families of observers will be referred to as the Hubble frame (or the CMB frame) and the “tilted” frame respectively. The 3-velocity vectors of these observers, \( \mathbf{u}_a \) and \( \mathbf{\tilde{u}}_a \) respectively, are related by the familiar Galilean transformation

\[
\mathbf{\tilde{u}}_a = \mathbf{u}_a + \mathbf{\tilde{v}}_a ,
\]

where \( \mathbf{\tilde{v}}_a \) is the peculiar velocity of the tilted observers relative to their CMB counterparts.\(^{10}\) Since \( \mathbf{\tilde{v}}_a \)-field is the local peculiar velocity, \( \mathbf{\tilde{u}}_a \) represents the total velocity of observers residing inside the bulk-flow domain.

The \( \mathbf{\tilde{u}}_a \) and \( \mathbf{u}_a \) fields also introduce a set of convective-derivative operators along their corresponding (spatial) directions. In particular, following § 9.1, we define

\[
\begin{align*}
\mathcal{\dot{L}} & = \partial_t + \mathbf{\tilde{u}}^\alpha \partial_\alpha \\
\mathcal{\dot{L}}_H & = \partial_t + \mathbf{u}^\alpha \partial_\alpha ,
\end{align*}
\]

as the convective derivatives in the tilted and the Hubble frames respectively.

10.2 The peculiar velocity field

The information regarding the kinematics of the aforementioned two groups of observers follow by splitting their gradients along the lines of decomposition (9.2.3), for the \( \mathbf{u}_a \)-field and an exactly analogous for the gradient of its tilted counterpart (\( \mathbf{\tilde{u}}_a \)). At the same time and in analogy with its relativistic analogue (see (6.3.2) in § 6.3 earlier), the gradient of the Newtonian

\(^{10}\)Newtonian studies of peculiar flows typically use physical \( (r^\alpha) \) and comoving \( (x^\alpha) \) coordinates, with \( r^\alpha = a \tilde{x}^\alpha \). The time-derivative of the latter gives \( \dot{v}_t = v_H + v_p \), where \( \dot{v}_t = \dot{r}^\alpha , v_H = H r^\alpha \) and \( v_p = a \dot{x}^\alpha \) are the total, the Hubble and the peculiar velocities respectively. On an FRW background, the above relation coincides with (10.1.1).
peculiar velocity field splits into its irreducible components according to
\[ \partial_\beta \tilde{v}_\alpha = \zeta_{\alpha\beta} + \tilde{\omega}_{\alpha\beta} + \frac{1}{3} \tilde{\vartheta}_{\alpha\beta}, \tag{10.2.1} \]
with \( \zeta_{\alpha\beta} = \partial_\beta \tilde{v}_\alpha, \tilde{\omega}_{\alpha\beta} = \partial_\beta \tilde{v}_\alpha \) and \( \tilde{\vartheta} = \partial^\alpha \tilde{v}_\alpha \). These respectively represent the peculiar shear, the peculiar vorticity and the peculiar volume expansion/contraction.

The three velocity fields defined above are related by a set of formulae, which depend on the velocity of their relative motion. When dealing with slow peculiar motions, the nonlinear Newtonian relations are formally identical to their linear relativistic counterparts (see Eq. (7.1.1) in § 7.1). More specifically, it is straightforward to show that the Newtonian expressions are
\[ \tilde{\vartheta}_\alpha = \vartheta_\alpha + \tilde{\omega}_\alpha \]
\[ \tilde{\omega}_\alpha = \tilde{\omega}_\alpha \]
\[ \tilde{\vartheta}_\alpha = \vartheta_\alpha + \tilde{\omega}_\alpha \]
(10.2.2)
by construction. The above are supplemented by the relation between the inertial acceleration vectors measured in the two frames, namely by [26]
\[ \ddot{v}_\alpha = \dot{v}_\alpha + \frac{1}{3} \Theta \tilde{v}_\alpha + \left( \tilde{\sigma}_{\alpha\beta} + \tilde{\omega}_{\alpha\beta} \right) \tilde{v}_\beta. \tag{10.2.3} \]
The latter is nonlinear as well, which means that it can be linearised around any (Newtonian) cosmological background.

10.3 The linear regime

Let us confine to a perturbed, almost-FRW, cosmological model containing pressure-free matter and focus on the three volume scalars. We ensure the linear nature of our analysis, by demanding that \( \tilde{\vartheta}/\Theta \ll 1 \) at all times. Following (10.2.2a), this means that bulk flows with \( \tilde{\vartheta} > 0 \) will have \( \tilde{\Theta} \gtrsim \Theta \) and it will expand slightly faster than the background universe. Observers inside contracting bulk flows, on the other hand, will have \( \tilde{\Theta} \lesssim \Theta \) and experience slightly slower expansion.\(^{11}\) In either case, the agent solely responsible for these differences is the peculiar flow. Clearly, however, the relative-motion effect on the expansion is local and only affects the neighbourhood of the bulk-flow domain.

The different expansion rates between the idealised Hubble-flow observers and their tilted partners seen in Eq. (10.2.2a), imply that relative motion will induce differences in the associated deceleration/acceleration rates as well. Indeed, by differentiating (10.1.1) with respect to time, we obtain
\[ \tilde{\Theta}' = \tilde{\Theta} + \tilde{\vartheta}' \tag{10.3.1} \]
to linear order. Recall that primes and overdots indicate convective derivatives in the “tilted” and the Hubble frames respectively. Put another way \( \tilde{\Theta}' = \partial_\beta \tilde{\Theta} + \tilde{w}^\beta \partial_\beta \tilde{\Theta}, \tilde{\Theta} = \partial_\beta \Theta + w^\beta \partial_\beta \Theta \)
\(^{11}\) The volume scalar is related to the Hubble parameter, since \( \Theta = 3H \) and \( \tilde{\Theta} = 3\tilde{H} \), with \( H \) and \( \tilde{H} \) being the Hubble parameters in the CMB and the tilted frames respectively. Then, Eq. (10.2.2a) also reads \( \tilde{H} = H + \tilde{\vartheta}/3. \)
and \( \ddot{\vartheta} = \partial_t \dot{\vartheta} + \dot{u}^\beta \partial_\beta \dot{\vartheta} \). Also, the sign and the value of \( \ddot{\vartheta} \) determine the local (volume) deceleration/acceleration of the peculiar flow itself.

As in the relativistic case, the differences between the volume expansion scalars and those between their convective derivatives, seen in (10.2.2a) and (10.3.1), suggest that the associated deceleration parameters should differ as well. Expressed in the tilted and the Hubble frames the latter are defined as

\[
\ddot{q} = -\left(1 + \frac{3\ddot{\Theta}'}{\Theta^2}\right) \quad \text{and} \quad q = -\left(1 + \frac{3\dot{\Theta}}{\Theta^2}\right),
\]

(10.3.2)

respectively. Putting aside the differences in the definitions of the kinematic variables involved, the above are formally identical to their relativistic analogues (see (8.1.2) in §8.1). Solving the above for \( \ddot{\Theta}' \) and \( \dot{\Theta} \), substituting the resulting expressions into Eq. (10.3.1) and then linearising leads to

\[
1 + \ddot{q} = (1 + q) \left(1 - \frac{2}{3} \frac{\ddot{\vartheta}}{H}\right) + \frac{1}{2} \frac{\ddot{\vartheta}'}{H}.
\]

(10.3.3)

The above is formally identical to its relativistic analogue (see expression (8.1.4) in §8.1), provided the aforementioned differences in the definitions are accounted for. In what follows, we will employ Eq. (10.3.3) to evaluate the difference between \( \ddot{q} \) and \( q \) within the boundaries of Newtonian physics.

10.4 The role of the gravitational field

In Newtonian theory, gravity is a force triggered by spatial variations in the gravitational potential. The latter couples to the density of the matter via Poisson’s equation. Next, we will discuss the implications of the Newtonian approach for the linear local expansion rate of the bulk-flow observers and for the related acceleration/deceleration rate.

The linear nature of our study demands that \( \dot{\vartheta}/H \ll 1 \) always, which in turn ensures that the effect of this ratio on the deceleration parameter in (10.3.3) is negligible. Then, just like in the relativistic study (see §8.1 earlier), the latter reduces to

\[
\ddot{q} = q + \frac{1}{2} \frac{\ddot{\vartheta}}{H} = q - \frac{1}{3} \frac{\ddot{\vartheta}'}{H^2},
\]

(10.4.1)

given that \( \dot{H} = -3H^2/2 \) in the Newtonian background. This leaves the peculiar volume deceleration/acceleration (i.e. the scalar \( \ddot{\vartheta}' \)) as the only realistic possibility for a measurable relative-motion effect on \( \ddot{q} \). Taking the convective derivative of \( \dot{\vartheta} \), in the tilted frame, recalling that \( \dot{\vartheta} = \partial^a \dot{v}_a \) and keeping up to linear-order terms, we arrive at

\[
\ddot{\vartheta}' = -H \ddot{\vartheta} + \partial^a \dot{v}_a',
\]

(10.4.2)

to first approximation. The above, which is formally identical to its relativistic counterpart (compare to expression (7.2.1) in §7.2), reveals how the background expansion slows down the
growth of $\tilde{v}$, by acting as (effective) friction. Also, not surprisingly, the temporal evolution of the peculiar volume scalar depends on the time-derivative of the peculiar velocity. Therefore, so far, the Newtonian analysis has proceeded in close parallel with the relativistic. Indeed, apart from differences in a number of definitions, the Newtonian equations are practically indistinguishable from their relativistic analogues.

The two approaches start to diverge when the gravitational field comes into play. In Einstein’s theory gravity is not a force, but a manifestation of the non-Euclidean geometry of the spacetime, namely of spacetime curvature. The latter results from the presence of matter, the contribution of which is encoded in the local energy-momentum tensor. When studying large-scale peculiar motions, the key contribution comes from the matter flux associated with the bulk flow (see § 7).

In Newtonian physics, on the other hand, the geometry is always Euclidean and gravity is a force triggered by gradients in the associated potential. As a result the corresponding energy and momentum conservation laws differ considerably from their relativistic counterparts. In particular, linearised around an Friedmann universe, the Newtonian Navier-Stokes formula leads to the following propagation equation for the peculiar velocity field [26,27]

$$\tilde{v}_a' = H\tilde{v}_a - \partial_a \Phi,$$  

(10.4.3)

which ensures that $\partial_a \Phi$ is the sole source of linear peculiar-velocity perturbations. Note that the above is identical to the expressions obtained in typical (though not necessarily covariant) studies (e.g. see [93,94]), provided the latter are written in physical rather than comoving coordinates. Nevertheless, Eq. (10.4.3) differs considerably from expression (7.2.3) in § 7.2, which monitors the linear evolution of the peculiar velocity field within the framework of Einstein’s theory. As a result, one should expect the Newtonian (linear) relative-motion effects on the deceleration parameter to differ considerably as well.

10.5 Newtonian relative-motion effects on $\tilde{q}$

As in the relativistic case, the Newtonian relative-motion effects on $\tilde{q}$, also depend on the divergence of $\tilde{v}'_a$. This time, however, the latter satisfies Eq. (10.4.3), the divergence of which involves the Laplacian of the gravitational potential. Adopting the “Jeans’ swindle” (e.g. see [95]), the (perturbed) gravitational potential couples to matter perturbations by means of the linearised Poisson equation

$$\partial^2 \Phi = \frac{1}{2} \kappa \rho \delta,$$  

(10.5.1)

where $\delta = \delta \rho / \rho$ is the familiar density contrast. Substituting (10.4.3) back into the right-hand side of Eq. (10.4.2) and then using (10.5.1), leads to

$$\tilde{q}' = -2H\tilde{q} - \frac{1}{2} \kappa \rho \delta.$$  

(10.5.2)
This result shows that perturbations in the density distribution can (in principle at least) force bulk flows to contract or expand locally. As expected, overdensities in the matter distribution (i.e. those with $\delta > 0$) cause local contraction, whereas underdensities lead to expansion.

Our final step is to combine Eqs. (10.4.1) and (10.5.2). In particular, recalling that $H^2 = \kappa \rho / 3$ in the (Newtonian) FRW background and keeping up to first-order terms, we arrive at

$$\tilde{q} \simeq q + \frac{1}{2} \delta.$$  \hspace{1cm} (10.5.3)

The above is the linear Newtonian relation between the deceleration parameter ($\tilde{q}$) measured in the rest-frame of typical observers in the universe and the deceleration parameter ($q$) measured by their idealised counterparts following the Hubble frame. Assuming that $q$ is of order unity, expression (10.5.3) guarantees that $\tilde{q} \simeq q$ throughout the linear regime, during which $\delta \ll 1$. Therefore, within linear Newtonian cosmological perturbation theory, the relative-motion effects cannot change the (local) sign of the deceleration parameter. Put another way, the Newtonian effects cannot make a decelerating universe appear accelerating and vice-versa. \hspace{1cm} \(12\)

\hspace{1cm} \(12\) Taken at face value, relation (10.5.3) seems to suggest that $\tilde{q}$ could take negative values in low-density domains that expand faster than the background universe, namely in voids with $\delta \to -1$ and $\dot{\delta} > 0$. In that case $\tilde{q}$ can become marginally negative, even when $q \simeq 1/2$. Having said that, one should be very cautious before applying linear results to nonlinear structures, like the large-scale voids. An alternative (also unlikely) possibility occurs when $0 < q \ll 1$, in which case $\tilde{q}$ turns negative when $\delta < 0$ and $|\delta| > q$.  

60
Bulk peculiar flows appear to be the norm in our universe. A series of surveys have repeatedly reported the presence of such coherent large-scale motions [7–15]. On theoretical grounds as well, peculiar flows are the inevitable byproduct of the ongoing process of structure formation [19,20]. There are still open issues however. The observations, for example, seem to agree on the direction of the aforementioned bulk motions, but not on their scale and on the magnitude of the associated velocities. There have also been reports of peculiar motions much larger and much faster than it is generally anticipated [16–18]. From the theoretical perspective too, most cosmological studies bypass peculiar motions and the few that account for them typically operate within the limits of Newtonian physics and take the viewpoint of the idealised Hubble-flow observers, rather than that of their real Bulk-flow counterparts.

Relative motions have been known to trigger apparent effects, which an unsuspecting observer may falsely interpret as “reality” and the history of astronomy is rife with such examples. With these in mind and given that the reported bulk flows extend out to several hundreds of Mpc, it is conceivable that the observers’ peculiar motion may have “contaminated” their cosmological data. It is straightforward to show, for example, that the local expansion and deceleration/acceleration rates, as measured inside the bulk-flow domains, are generally different from those of the actual universe. Moreover, the aforementioned differences are not “real”, but apparent and entirely due to relative-motion effects (see Eqs. (7.1.1a) and (8.1.1) respectively). The emerging question is then whether (and under what conditions) these differences can become large enough to cause a serious misinterpretation of the incoming data and thus affect the way the observers involved understand the universe their happen to live in. This was the aim of the investigation that led, among others, to the present Thesis.

We have employed linear relativistic cosmological perturbation theory to study the effects of large-scale peculiar motions on the deceleration parameter of the universe, as measured by observers living inside these bulk flows. Given that the difference between \( \dot{q} \) and \( q \), at the linear perturbative level, depends on \( \dot{\varphi} \) (see expression. (8.1.5) in § 8.1), we first focused on the time evolution of the local volume expansion/contraction scalar. This meant deriving the Raychaudhuri equation of the peculiar flow, which monitors the local expansion/contraction of the bulk-flow domain [79]. Linearising the latter formula around an FRW model with dust, we found that it introduces a scale-dependence to the difference between the deceleration parameter measured in the tilted frame and its Hubble-flow counterpart (see [30], as well as Eq. (8.2.3) in § 8.2 here). As a result, on large enough scales (close and beyond the Hubble horizon), the value of \( \dot{q} \) essentially coincides with that of \( q \). This was not unexpected, since the impact of the observer’s peculiar motion is believed to fade away as one moves out to progressively larger wavelengths. On scales well inside the Hubble horizon, however, the relative-motion effect on the locally measured deceleration parameter (\( \dot{q} \)) can be large.

Confining to subhorizon scales, we found that peculiar flows introduce a characteristic threshold, termed the peculiar Jeans length (\( \lambda_P \) – see [31,27] and also Eq. (8.3.1) in § 8.3 here),
below which the local kinematics are dominated by relative-motion effects, rather than by the background Hubble expansion. The size of the aforementioned critical scale, which is closely analogous to the familiar Jeans length (also resulting from linear perturbation theory [20,6]), is determined by the velocity of the drifting domain. Using data reported by some of the recent bulk-flow surveys, we have estimated that the peculiar Jeans length ($\lambda_P$) typically varies between few hundred and several hundred Mpc (see Table 1 in § 8.5), although it is conceivable that projection effects could extend its range even further. On scales smaller than $\lambda_P$, the cosmological data can be drastically contaminated by relative-motion effects. In particular, the value of the deceleration parameter inside slightly expanding bulk flows can significantly exceed the one measured in the Hubble frame of the mean universal expansion. The effect is reversed inside contracting bulk motions, where the deceleration parameter drops well below its Hubble-flow value, while in some cases it can even become negative (on scales smaller than the associated peculiar Jeans length – see Figs. 3 and 4). In the latter case, which is also the most intriguing one, the peculiar Jeans length also marks the transition scale ($\lambda_T$), where the deceleration parameter changes from sign [31,27].

The relative-motion effects discussed so far are purely relativistic in nature, with no close Newtonian analogue. The reason is the fundamental differences between the two theories in the way they treat, space, time and (most importantly for our purposes) the gravitational field itself. More specifically, according to Newton, gravity is a force that is triggered by spatial variations in the gravitational potential. Also, only the density of the matter contributes to the gravitational field via Poisson’s equation. In relativity, on the other hand, gravity is the manifestation of spacetime curvature, which itself is triggered by the presence of matter. Here, however, it is not only the density that contributes the the local energy-momentum tensor. The pressure, both the isotropic and the anisotropic, as well as any energy flux that may be present also have their own separate input to the gravitational field. When studying the effects of peculiar motions, the flux-contribution plays the pivotal role, since then there is always a nonzero-energy flux vector, even at the linear level and entirely due to relative-motion effects [25,26]. This purely general relativistic effect feeds into the conservation laws, changes the formulae monitoring the evolution of the peculiar velocity field and eventually alters its effects on the local kinematic, including the locally measured deceleration parameter (see [31,27], as well as § 8 and § 10 here). In the absence of any energy-flux contribution to the gravitational field, the Newtonian analysis leads to entirely different results. More specifically, within the limits of Newtonian gravity, the relative-motion effects on the deceleration parameter are negligible for all practical purposes. The values of $\tilde{q}$ and $q$, as measured in the bulk-flow and the Hubble-flow frames respectively, essentially coincide [27].

---

13 Observers living inside expanding peculiar flows may misinterpret their locally faster expansion rate as over-deceleration of the surrounding universe. In contrast, those inside slightly contracting bulk motions may misinterpret their slower local expansion as global under-deceleration, or even as universal acceleration in some cases. Intuitively speaking, one could imagine of these observers as passengers sitting at the back of a car driving down a motorway. Then, if their car’s speed drops below the average, the unsuspecting passengers could be mislead to believe that the rest of the vehicles have accelerated away (and vice versa).
The overall impact of the relative motion on the deceleration parameter depends on the scale and the speed of the bulk-flow domain. Thus, the larger the scale the weaker the effect, given that peculiar velocities decay with increasing wavelength. Put another way, the impact of the relative motion keeps growing as one moves down to progressively smaller lengths and closer to the observer (see Table 1 for representative values). More specifically, the scale/redshift distribution of the deceleration parameter, as measured by the bulk-flow observers, follows from Eq. (8.4.1) and it is depicted in Fig. 3 (see § 8.4 earlier). Even when we consider the simplest scenario, with \( \vec{\theta} = \text{const.} \) throughout the bulk-flow domain, the profile of the solid curve in the aforementioned figure closely resembles those of the deceleration parameters reconstructed from the supernovae data in [80–83]. According to Eq. (8.4.1) and the solid curve seen in Fig. 3, the local deceleration parameter approaches its Hubble-frame value on large enough scales/redshifts and turns negative closer to the observer. This distinct behaviour, which is in qualitative agreement with the observations, is also a phenomenological prediction of the bulk-flow scenario. Another prediction is an apparent (Doppler-like) dipole in the sky-distribution of \( \dot{q} \), triggered by the direction of the observer’s relative motion. The deceleration parameter should take more negative values in a given direction in the celestial sphere and equally less negative in the antipodal. Put another way, the bulk-flow observers should “see” their universe accelerate faster along one direction in the sky and equally slower along the opposite. Moreover, the dipolar axis in the \( \dot{q} \)-distribution should not lie far from its CMB counterpart, assuming that both are triggered by relative-motion effects. Over the last ten years or so, there have been reports in the literature that such a preferred axis may actually exist in the supernovae data [32–37]. Intriguingly, an analogous dipolar anisotropy, this time in the sky-distribution of the Hubble parameter, was also recently reported [89,90].

Based on the above, it appears theoretically plausible that the recent accelerated expansion of the universe may be nothing more than a mere illusion caused by our peculiar motion relative to the smooth Hubble flow. The attractive features of such a scenario are that it operates within standard general relativity and within the linear regime of a perturbed Einstein-de Sitter universe. There is no need for introducing any new physics, or for appealing to exotic forms of matter. The inferred acceleration is not a global event, but a local artefact of the observers bulk peculiar flow. This guarantees that there is no “coincidence problem” either, since the transition from deceleration to (apparent) acceleration is a byproduct of the ongoing structure-formation process and occurs naturally at the transition length \( \lambda_T \). Moreover, the profile of the predicted scale/redshift-distribution of \( \dot{q} \) appears to agree with those reconstructed from the supernovae data. There are caveats as well of course. To this point, the transition scale appears smaller than the one typically inferred from the reconstructed data, although it is conceivable that introducing a scale-dependence on \( \vec{\theta} \) may reconcile the difference. Also, the numerics depend on the peculiar-velocity divergence (\( \vec{\theta} \)) are still very difficult to extract from the observations. Therefore, at this stage, the signatures of the bulk-flow scenario, namely the predicted scale/redshift -distribution of the deceleration parameter and an apparent (Doppler-like) dipole in its sky-distribution, are still qualitative in nature.

It should also be noted that, provided there is no natural bias in favour of expanding, or contracting, bulk flows on cosmologically relevant scales, namely those with \( \lambda \geq 100 \text{ Mpc} \), the chances of an observer residing inside either of them should be close to 50%. Then, according
to the bulk-flow scenario, nearly half of the observers in the universe may think that their cosmos is over-decelerated, with \( \dot{q} > q \). The other half, on the other hand, will measure \( \dot{q} < q \) in their own rest-frame and they could be misled to believe that the universal expansion is under-decelerated. In fact, some of the latter observers may even measure \( \dot{q} < 0 \), in which case they could be erroneously forced to think that the whole universe has recently entered a phase of accelerated expansion.
A Appendix

A.1 Transformations under a 4-velocity boost

This section of the appendix provides the complete set of the relativistic transformation laws between two relatively moving frames. These formulae, which were first given in [68], hold in a general spacetime and for arbitrary peculiar velocities. When linearised around an FRW background model, the expressions given below reduce to the relations employed in § 7.1.

Consider an observer moving with 4-velocity \( \tilde{u}_a \) relative to the reference \( u_a \)-frame. The two 4-velocity fields are related by the Lorentz transformation

\[
\tilde{u}_a = \tilde{\gamma}(u_a + \tilde{v}_a),
\]  

(A.1)

where \( \tilde{\gamma} = (1 - \tilde{v}^2)^{-1/2} \) is the associated boost factor and \( \tilde{v}_a \) is the ‘peculiar’ velocity of the \( \tilde{u}_a \)-field (with \( u_a \tilde{v}^a = 0 \)). Note that for non-relativistic peculiar motions \( \tilde{v}^2 \ll 1 \) and \( \tilde{\gamma} \simeq 1 \).

The projection tensors (\( \tilde{h}_{ab} \) and \( h_{ab} \)) corresponding to the aforementioned 4-velocities and the the Levi-Civita tensors of the associated 3-D hypersurfaces are related by

\[
\tilde{h}_{ab} = h_{ab} + \frac{2}{\tilde{\gamma}} v^2 u_a u_b + 2 u_{(a} v_{b)} + v_a v_b,
\]  

(A.2)

and

\[
\tilde{\varepsilon}_{abc} = \gamma \varepsilon_{abc} + \gamma \left(2 u_{[a} \varepsilon_{b]cd} + u_c \varepsilon_{abd}\right) v^d,
\]  

(A.3)

respectively. Recall that \( \varepsilon_{abc} = \eta_{abcd} u^d \) and \( \tilde{\varepsilon}_{abc} = \eta_{abcd} \tilde{u}^d \), with \( \eta_{abcd} \) representing the Levi-Civita tensor of the spacetime.

Analogous nonlinear algebraic relations also hold between the irreducible kinematic variables measured in the two frames. In particular, we have the relation

\[
\tilde{\Theta} = \gamma \Theta + \gamma (D_a v^a + A^a v_a) + \gamma^3 W,
\]  

(A.4)

between the expansion/contraction scalar. More involved are the expressions relating the shear and the vorticity tensors, given by
\[ \tilde{\sigma}_{ab} = \gamma \sigma_{ab} + \gamma \left(1 + \gamma^2\right) u_{(a}\sigma_{b)c}v^c + \gamma^2 A_{(a} \left[v_{b)} + v^2 u_{b)} \right) \\
+ \gamma D_{(a}v_{b)} - \frac{1}{3} h_{ab} \left[A_c v^c + \gamma^2 \left(W - \dot{v}_c v^c\right)\right] \\
+ \gamma^3 u_{ab} \left[ \sigma_{cd}v^c v^d + \frac{2}{3} v^2 A_c v^c - v^c v^d D_{(c} v_{d)} + \left(\gamma^4 - \frac{1}{3} v^2 \gamma^2 - 1\right) W \right] \\
+ \gamma^3 u_{(a} v_{b)} \left[A_c v^c + \sigma_{cd}v^c v^d - \dot{v}_c v^c + 2\gamma^2 \left(\gamma^2 - \frac{1}{3}\right) W \right] \\
+ \frac{1}{3} \left[ \gamma^3 v_{a} v_{b} \left[D_{(c} v^c - A_c v^c + \gamma^2 \left(3\gamma^2 - 1\right) W \right] + \gamma^3 v_{(a} \dot{v}_{b)} + v^2 \gamma^3 u_{(a} \dot{v}_{b)\right]} \\
+ \gamma^3 v_{(a} \sigma_{b)c} v^c - \gamma^3 \omega^b v^c \varepsilon_{bc(a} \left(v_{b)} + v^2 u_{b)} \right) + 2\gamma^3 v^c D_{(c} v_{a)} \left(v_{b)} + u_{b} \right) \] (A.5)

and

\[ \tilde{\omega}_a = \gamma^2 \left[ \left(1 - \frac{1}{2} v^2\right) \omega_a - \frac{1}{2} \text{curl} v_a + \frac{1}{2} v_{b} \left(2\omega^b - \text{curl} v^b\right) u_{a} + \frac{1}{2} v_{b} \omega^b v_a \right] \\
+ \frac{1}{2} \varepsilon_{abc} A^b v^c + \frac{1}{2} \varepsilon_{abc} \varepsilon_{b} v^c a + \frac{1}{2} \varepsilon_{abc} \sigma^b_{d} v^c v^d \] (A.6)

respectively. Finally, we also have

\[ \tilde{A}_{a} = \gamma^2 A_{a} + \gamma^2 \left[ \dot{v}_{(a)} + \frac{1}{3} \Theta v_{a} + \sigma_{ab} \varepsilon^{b} - \varepsilon_{abc} \omega^c v^c + \left(\frac{1}{3} \Theta v^2 + A^b v_{b} + \sigma_{bc} \varepsilon^b v^c\right) u_{a} \right] \\
+ \frac{1}{3} \left( D_{b} v^b \right) v_{a} + \frac{1}{2} \varepsilon_{abc} \varepsilon_{b} v^c + v^b D_{(b} v_{a)} \right] + \gamma^4 W(u_{a} + v_{a}) \] (A.7)

between the 4-acceleration vector. Note that, in deriving Eqs. (A.2)-(A.7), we have used the auxiliary variable

\[ W \equiv \dot{v}_c v^c + \frac{1}{3} v^2 D_c v^c + v^c v^d D_{(c} v_{d)}. \] (A.8)

Similarly, when moving from the \( u_a \)-frame to its tilded counterpart, the energy density and the (isotropic) pressure of matter transform according to the nonlinear laws

\[ \tilde{\rho} = \rho + \gamma^2 \left[v^2 (\rho + p) - 2q_a v^a + \pi_{ab} v^a v^b\right] \] (A.9)

and

\[ \tilde{p} = p + \frac{1}{3} \gamma^2 \left[v^2 (\rho + p) - 2q_a v^a + \pi_{ab} v^a v^b\right], \] (A.10)

respectively. At the same time, the relations between the energy flux vectors and the viscosity tensors are

\[ \tilde{q}_a = \gamma q_a - \gamma \pi_{ab} v^b - \gamma^3 \left[(\rho + p) - 2q_b v^b + \pi_{bc} v^b v^c\right] v_a \\
- \gamma^3 \left[v^2 (\rho + p) - (1 + v^2) q_b v^b + \pi_{bc} v^b v^c\right] u_a \] (A.11)

and

66
\[
\tilde{\pi}_{ab} = \pi_{ab} + 2\gamma^2 v^c \pi_{c(a} \{ u_b \} + v_{b)} - 2v^2 \gamma^2 q_{(a} u_b) - 2\gamma^2 q_{(a} v_b) \\
- \frac{1}{3} \gamma^2 \left[ v^2(\rho + p) + \pi_{cd} v^c v^d \right] h_{ab} \\
+ \frac{1}{3} \gamma^4 \left[ 2v^4(\rho + p) - 4v^2 q_c v^c + (3 - v^2) \pi_{cd} v^c v^d \right] u_a u_b \\
+ \frac{2}{3} \gamma^4 \left[ 2v^2(\rho + p) - (1 + 3v^2) q_c v^c + 2\pi_{cd} v^c v^d \right] u_{(a} v_{b)} \\
+ \frac{1}{3} \gamma^4 \left[ (3 - v^2)(\rho + p) - 4q_c v^c + 2\pi_{cd} v^c v^d \right] v_a v_b . \tag{A.12}
\]

As with the kinematic variables, the linearised versions of Eqs. (A.9)-(A.12) were also used in § 7.1.

Finally, for completeness, we provide the nonlinear transformation laws between the electric and the magnetic components of the Weyl field. The latter are respectively given by

\[
\tilde{E}_{ab} = \gamma^2 \left\{ (1 + v^2) E_{ab} + v^c \left[ 2\varepsilon_{cd(a} H_{b)}^d + 2E_{c(a} u_b \right] \\
+ (u_a u_b + h_{ab}) E_{cd} v^d - 2E_{c(a} v_{b)} + 2u_{(a} \varepsilon_{b)cd} H^{de} v_e \right\} \tag{A.13}
\]

and

\[
\tilde{H}_{ab} = \gamma^2 \left\{ (1 + v^2) H_{ab} + v^c \left[ -2\varepsilon_{cd(a} E_{b)}^d + 2H_{c(a} u_b \right] \\
+ (u_a u_b + h_{ab}) H_{cd} v^d - 2H_{c(a} v_{b)} - 2u_{(a} \varepsilon_{b)cd} E^{de} v_e \right\} . \tag{A.14}
\]

A.2 Covariant commutation laws

According to definition (2.1.2a), the orthogonally projected covariant derivative operator satisfies the condition $D_a h_{bc} = 0$. This means that we can use $h_{ab}$ to raise and lower indices in equations acted upon by this operator. Following Frobenius' theorem, however, rotating spaces do not possess integrable 3-D submanifolds (e.g. see [69,70]). Therefore, the $D_a$-operator does not always satisfy the usual commutation laws (see below and also [75]).

When acting on a scalar quantity the orthogonally projected covariant derivative operators commute according to

\[
D_{[a} D_{b]} f = -\omega_{ab} \dot{f} . \tag{A.1}
\]

The above is a purely relativistic result and underlines the different behaviour of rotating spacetimes within Einstein’s theory. Similarly, the commutation law for the orthogonally projected derivatives of spacelike vectors reads

\[
D_{[a} D_{b]} v_c = -\omega_{ab} \dot{v}_{(c)} + \frac{1}{2} R_{dca} v^d . \tag{A.2}
\]
where \( v_a u^a = 0 \) and \( R_{abcd} \) represents the Riemann tensor of the observer’s local rest-space. Finally, when dealing with orthogonally projected tensors, we have

\[
D_{[a} D_{b]} S_{cd} = -\omega_{ab} h_c^e h_d^f \dot{S}_{ef} + \frac{1}{2} (R_{ecba} S_{e d} + R_{edba} S_{e c}) ,
\]

(A.3)

with \( S_{ab} u^a = 0 = S_{ab} u^b \). Note that in the absence of rotation, \( R_{abcd} \) is the Riemann tensor of the (integrable) 3-D hypersurfaces orthogonal to the \( u_a \)-congruence For details on the definition, the symmetries and the key equations involving \( R_{abcd} \), the reader is referred to § 3.3. We also note that the above equations are fully nonlinear and hold at all perturbative levels.

In general relativity, time derivatives do not generally commute with their spacelike counterparts. For scalars, in particular, we have

\[
D_a \dot{f} - h_a^b (D_b f)^\cdot = -\dot{f} A_a + \frac{1}{3} \Theta D_a f + D_b f \left( \sigma^b_a + \omega^b_a \right) ,
\]

(A.4)

at all perturbative levels.

Assuming an FRW background, where all vectors and tensors vanish identically and \( R_{abc} = (K/a^2)(h_{ac} h_{bd} - h_{ad} h_{bc}) \), relations (A.2), (A.3) and (A.4) linearise to

\[
D_{[a} D_{b]} v_c = \frac{K}{2a^2} (h_{ac} v_b - h_{bc} v_a) ,
\]

(A.5)

\[
D_{[a} D_{b]} S_{cd} = \frac{K}{2a^2} (h_{ac} S_{bd} - h_{bc} S_{ad} + h_{ad} S_{cb} - h_{bd} S_{ca})
\]

(A.6)

and

\[
D_a \dot{f} - h_a^b (D_b f)^\cdot = -\dot{f} A_a + H D_a f ,
\]

(A.7)

respectively. In addition, also on an FRW background, the orthogonally projected gradient and the time derivative of the first-order vector \( v_a \) commute as

\[
a D_a \dot{v}_b = (a D_a v_b)^\cdot ,
\]

(A.8)

to linear order. Similarly, when dealing with first-order spacelike tensors, we have the following linear commutation law

\[
a D_a \dot{S}_{bc} = (a D_a S_{bc})^\cdot .
\]

(A.9)

### A.3 Scalar, vector and tensor modes

In the coordinate-based approach, perturbations are decomposed from the start into scalar, vector and tensor modes, using appropriate harmonics. The covariant approach does not depend on a priori splitting into harmonic modes and it is independent of any Fourier-type decomposition. Instead, all the perturbative quantities are described as spatial vectors \( V_a = V_{(a)} \) or as spatial, symmetric and trace-free rank-2 tensors \( S_{ab} = S_{(ab)} \).
The scalar modes are characterised by the fact that all vectors and tensors are generated by scalar potentials. For instance,

\[ V_a = D_a V \quad \text{and} \quad S_{ab} = D_{(a} D_{b)} S , \]

for some \( V, S \). This implies that \( \text{curl} \, V_a = 0 = \text{curl} \, S_{ab} \).

For vector modes, all vectors are transverse (solenoidal) and proportional to \( \omega_a \). Also, all tensors are generated by transverse vector potentials. Thus,

\[ D^a V_a = 0 \quad \text{and} \quad S_{ab} = D_{(a} S_{b)} , \]

where \( D^a S_a = 0 \). Vector modes are nonzero if and only if the vorticity is nonzero.

Tensor modes are characterised by the vanishing of all vectors and by the transverse traceless nature of all tensors. In other words,

\[ V_a = 0 \quad \text{and} \quad D^b S_{ab} = 0 . \]

This way no perturbative scalars or vectors can be formed.

We can expand these modes in harmonic basis functions (Fourier modes in the case \( K = 0 \)). For example, for scalar modes, the harmonics are time-independent eigenfunctions that satisfy the scalar Laplace-Beltrami equation. In other words, \( \dot{Q}^{(n)} = 0 \) and

\[ D_a^2 Q^{(n)} = - \left( \frac{n}{a} \right)^2 Q^{(n)} , \]

where \( n \) is the eigenvalue of the associated harmonic mode and \( D^2 = D^a D_a \). The latter takes continuous values when \( K = 0, -1 \) and discrete ones for \( K = +1 \). In particular, \( n = \nu \geq 0 \) when the 3-space has Euclidean geometry and \( n^2 = \nu^2 + 1 \geq 0 \) for hyperbolic spatial sections, with \( \nu \) representing the comoving wavenumber of the mode in all cases (e.g. see [96]). Supercurvature modes have \( \lambda = a/n > a \) and in open FRW models correspond to \( 0 < n^2 < 1 \). Those with \( n^2 > 1 \), on the other hand, span scales smaller than the curvature radius and are therefore termed subcurvature. Clearly, the \( n^2 = 1 \) threshold indicates the curvature scale, with \( \lambda = \lambda_K = a \). Note that, although they are often ignored, supercurvature modes are necessary if we want perturbations with correlations lengths bigger than the curvature radius [71]. Finally, when the 3-curvature is positive, \( n^2 = \nu (\nu + 2) \) and \( \nu = 1, 2, \ldots \).
References


